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School and college mathematics
WILLIAM L. DUREN, JR.

Some remarks on enrichment
IZAAK WIRSZUP

On the formulation of certain arithmetical questions
HERBERT J. CURTIS *and* KARL MENGER

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School and college mathematics[1]

WILLIAM L. DUREN, JR., *University of Virginia, Charlottesville, Virginia.*

A mathematician's thoughts on how to revise the high-school mathematics courses so as to minimize the "break" in the mathematics program at the beginning of the thirteenth year.

OUR PRESENT POSITION

WE HAVE WITNESSED in the past five years a remarkable change in the public attitude towards mathematics in the schools. Just a few years ago, and persisting to the present time in many localities, mathematics was comparatively poorly appreciated, neglected, and often attacked in the schools. Mathematics represented the old tyranny of the disciplines. It was hard to justify mathematics, and hard to get it taught. Mathematics had come to be regarded by many educational administrators as useless, if not detrimental, to the modern youth. It was due to go the way of the classics. Yet, somehow it did not. Too much of our technical civilization was dependent on mathematics for it simply to be wished away by educational theories. And it was too substantial a discipline, too full of ideas and real content for young people, to be replaced by any glittering new subjects which superficially seemed to be more practical.

Even so the mathematics teaching staff of the schools was allowed to run down in quantity and quality. Many misfits in the teaching profession, as well as football coaches, were given the unpopular job of teaching mathematics. The morale of the old guard that remained was low because the teachers and their subject were so poorly appreciated.

[1] Presented to National Council of Teachers of Mathematics, Milwaukee, April 14, 1956

But there has been a change in the public attitude. This change was not brought about by academic speech-making but by the fact that leaders in industry and government began to realize that our economic welfare and our military security were dependent upon maintaining an expanding staff of technically trained men. They began to tell the public the facts about the degeneration in the learning of science and mathematics in the schools and about the poor showing which an American youth makes in these skills as compared with a European, especially a Russian youth. They began to bid competitively for the short supply of trained young people, actively including "mathematicians" in their lists of wanted men along with engineers, physicists, and chemists. For mathematics itself, the development of computers vastly increased this demand.

At first there was a natural tendency to seek physicists and chemists from among those who were turned toward physics or chemistry by an inspiring high-school teacher of the subject. But then it began to be apparent that high-school mathematics was one of the most important training fields, if not *the* most important for scientists and engineers. It is probably more important than high-school science, and almost certainly more important than college mathematics, in setting up a career in science or engineering. It soon became apparent that the high school was the place to look for an increased supply of scientists. So, high-school mathematics

came rather suddenly back into the category of subjects, that are important for nearly everybody and for the nation as a whole.

However, the professional teachers of mathematics in the high schools and in the colleges were unprepared for this favorable event. The process of postponing the more advanced and difficult subjects had forced much of what was formerly grade-school mathematics into the high school and much of what was formerly high school mathematics into the college curriculum. This led to a breakdown in the old rigid sequence of standard courses so that as many as seven significantly different types of first-year college courses in mathematics were taught. Within the mass education system this resulted in chaos and in much duplication of course work, particularly by the abler students who got the worst of it. In particular, such courses as plane and solid geometry, college algebra, and analytic geometry had been watered down so as to be almost unrecognizable, and so that they were no longer fit for a curriculum resulting from a revived interest in good mathematical training. Solid geometry, to take only one example, was dead, and there was nothing to take its place. Even now, it will take years to construct a suitable subject to take the place of solid geometry and prepare for teaching it by mass education.

There is one thing that we must do in taking advantage of the renewed public support to reconstitute the genuine values of the old mathematics program. We must eliminate wasteful duplication and leave to the appropriate schools a significant share of the mathematics training which is theirs and theirs alone.

Division of responsibility between high school and college

This calls for a clear-cut division of responsibility in the teaching of mathematics between high school and college. My own opinions on this division of responsibility come from our work in the Committee on the Undergraduate Program of the Mathematical Association of America. They are as follows:

The normal beginning of college mathematics should be analytic geometry and calculus. Intermediate algebra, trigonometry, and a considerable amount of analytic geometry in the form of graphs should be the business of the high schools. Especially able students in high school, but only a few of them, should have the opportunity to take calculus, possibly receiving college credit for it. Conversely, some retarded students (few, I hope!) will need to take intermediate algebra in college, but this remedial mathematics should be understood as a high-school course and not a proper part of the degree program in college. The faster we can eliminate this duplicated service the better will be the morale in high-school mathematics.

In the division of responsibility, high-school mathematics ought not to be merely a preparation for college mathematics, dictated in form and content by college mathematicians. High-school mathematics ought to justify itself, to have its own values independent of preparation for college. Hence, in the division of responsibility, there should remain only as much of a common core of college preparation in the high-school curriculum as is absolutely essential. The valuable service of the university mathematicians will not be to demand that the high schools measure up to their standards of mathematical preparation. But the university mathematicians will necessarily be called upon to put together the basic mathematical content of the new course work for the schools, and to train the teachers. This leaves to the teachers of mathematics in the schools the most important job of all, that of capturing the interest and directing the enthusiasm of our youngsters so that we save three times as many of the ones competent for further studies of both scientific and humanistic nature as we now get.

THE GENERAL NATURE OF NEW SCHOOL MATHEMATICS

Before proceeding to propose some particular new high-school courses, I should like to speak a bit about the general nature of the "new high-school program in mathematics." Should it not contain the "modern" mathematics, Boolean algebra, perhaps, logic, set theory, digital analysis, computer programing? I rather think not, but it should still be concerned primarily with mathematical situations which are to be described by means of a finite number of the elementary field operations, or of geometric configurations. Should not the new program emphasize the humanistic values of mathematics? The historical approach? I think not, even though we mathematicians can be proud of the ancient humanistic heritage of mathematics, older than English by far, and older than History. The trouble is that appreciation cannot begin properly until the subject is understood, and to attempt to teach directly the meaning and importance of mathematics, forces the teacher into an external language so that he talks *about* mathematics to the exclusion of having the students talk and do mathematics itself. Logic? I doubt the value of the study of logic in the high schools for similar reasons and also because very, very few of the enthusiasts for logic in mathematics know any logic. It has not been proved yet that formal logic is in any way better than trigonometry as a mind trainer, and trigonometry has no overwhelming record of success in this field. Rather, we must try to develop the same old mathematical powers; the powers of synthesis and analysis, of making an abstraction rather than of learning the previously abstracted mathematical fact. Arithmetic, algebra, and geometry are still the best materials for the high-school course.

One aspect of the subject matter of any good course in mathematics remains important; the material must constitute a mathematical structure which follows in correct logical order from well-stated assumptions. The values of mathematics come from mathematics itself. Thus, to depart from the essential nature of mathematics and present disjointed topics for their value in applications, or for any other so-called "practical" reason, is to destroy the very thing you are trying to teach. Your course must have mathematical form and substance. Once you have this you can begin to tie on the motivations, the interpretations, and the applications. But the theory should determine the structure of your course.

To construct the courses which we now need in the high schools, we need to start with well-formed lines of mathematical theory chosen for their appropriateness. Given such a line of mathematical theory, and artful text writer can present, illustrate, and interpret the material so that it is more accessible to young minds. But when this editing and interpreting has gone through several successive textbooks, the mathematical content tends to diminish at each stage until there is little left but confusing talk. This is the state of our school textbooks today. By contrast, Euclid's geometry survived for two thousand years because it was a substantial line of mathematical theory in the first place, and, furthermore, was permitted to remain essentially in the original text form. Comparatively recently the textbook writers started to work on it and rapid decay has set in, already reaching the point where it is doubtful if the old life can ever be breathed back into it as a high-school subject.

So once in a while, we need to draw from higher mathematical scholarship a new line of theory and start over. Such a time has been long overdue. But if, in our haste to get new textbooks, we fall into the old textbook-writing rut, we will get only dreary new failures to reward the new enthusiasm. By the "textbook-writing rut," I mean the old process of selecting a list of topics which we wish to "cover,"

giving this to the skillful text writer who then sits down with ten old texts before him and dashes off a new creation to fit the specifications; a new creation which turns out in use to be neither new nor a creation and, in fact, not mathematics either.

A suggested new course

To illustrate concretely some of these general points, I should like to set forth one sequence of mathematical ideas which forms a good line of theory and which can be adapted in several ways to local conditions. Its topics include little that is new, but perhaps there exist no courses built upon such a line of theory. I think that it is appropriate for the twelfth year, perhaps for the eleventh year. It will include logarithms, geometric progressions, the compound interest law. It turns out to be only a short step from calculus on the one hand and from the mathematics of business and agriculture on the other. I shall present it as an outline. The mathematical details were fairly carefully worked out in *Universal Mathematics, Part I*[2]

Graphs, rates, and sums

Mathematical Outline

0. Assume the algebraic properties of real numbers and order properties as points on a line.
1. Intervals of numbers
2. Partitions of intervals
   a. Decimal and binary representation
3. Co-ordinate systems
   a. Algebraic and geometric analogies
4. Graphs of relations
   a. Linear relations
   b. Inequalities
5. Functional relations
   a. Graphs
   b. Graphical operations with functions
   c. Polynomials
   d. Maxima and minima
   e. Solution of equations
6. Rates
   a. Divided differences
   b. Convexity
   c. Algebraic operations with divided differences
7. Riemann sums
   a. Area interpretation
   b. Algebraic properties
   c. Step graphs and roof graphs
   d. The fundamental theorem of finite calculus
   e. Simpson's rule
8. Compound interest law
   a. Geometric progressions
   b. Exponential growth
   c. Logarithms
   d. Logarithmic graphs
   e. Inverse rate problems
9. Difference equations

This is essentially a course in the calculus of finite differences. It does not involve the idea of a limit, but the techniques require only a finite number of the arithmetic operations. One should modify the notation of finite differences so as to suppress the index machinery and make the divided difference notation and the sum notations closely resemble the simple notations for derivative and integral. The algebraic operations, interpretations, and applications should be pressed much farther than they are ordinarily. Ordinarily, a great many of the ideas are reserved for calculus, but there is no real reason to do this. By adroit use of notation and suitable choice of limit theory, one can make it a very short transitional step from this material to the calculus of polynomials, exponentials, logarithms, and general powers.

On the other hand this is the essential mathematics needed for courses in Business Mathematics or for courses in Agricultural Mathematics, so that in practical adaptation it might form a later substitute for what the old eighth-grade arithmetic was supposed to achieve in its original conception, namely, a review of the mathematical skills previously learned,

[2] *Universal Mathematics, Part I.* (Lawrence, Kansas, University of Kansas, 1954).

with applications to business and industry, in the last year of general education. By this choice of mathematical theory, the discrepancy between this practical (and terminal) adaptation of the mathematical material and the advanced (calculus) adaptation is lessened. This type of course might salvage college eligibility for many students who originally thought they were not going to college. It is attractive in the light of the democratic ideal of avoiding the sharp dichotomy which classifies one man finally as college material and another as not college material. And it is attractive also in its provision of a natural way for the more gifted and ambitious to move ahead without such an autocratic classification.

Have you read?

BERNSTEIN, ALLEN, "A Study of Remedial Arithmetic Conducted With Ninth Grade Students," *School Science and Mathematics*, January 1956, pp. 25–31 and June 1956, pp. 429–437.

We will never know too much about our students' problems in mathematics. This is a report of a study of difficulties stemming from instructional deficiencies, low achievement, test taking difficulties and emotional problems. Some of the mathematical difficulties studied were: borrowing; in subtraction, decimal placement difficulty, operations with zero, inverting of fractions, extreme slowness, extreme sloppiness and others. You will be interested in reading of some of the case studies and of the clinical approaches used to solve particular situations. For example, there is the case of Lela, who had difficulties with decimals, fractions, zeros and denominate numbers; after several days work in which these specific problems were studied she made almost a perfect score on the test. This study might well be a pattern we could use in trying to help our students.

BRONOWSKI, J., "The Educated Man in 1984," *The Advancement of Science*, Published by the British Association for the Advancement of Science, December 1955, pp. 306–310.

Have you ever considered that every class that you teach will be using what you are teaching 20 years later? Look back 20 years and note the changes that have been made. Are you preparing your students for conditions as they will exist 20 years from now? This article attempts to set up some guide posts to insure the future.

It points out, among other things, that science is as much a cultural subject of the future as Chaucer or Mozart; that the voter cannot be intelligent if he doesn't know the potential of science.

To do all this, the author believes that mathematics must be taught as a living language, with emphasis on relationships. The sciences need to be taught as an evolution of knowledge not as a collection of facts. The ideas of probability and statistics are parts of this scientific structure.

You will be interested in the author's contention that science is not the province of the gifted few only, but the rightful heritage of the total group. This article will be good for your students to read because it is about their future that the author is concerned.

WEAVER, WARREN, "Lewis Carroll: Mathematician," *Scientific American*, April 1956, pp. 116–128.

You, and probably your students also, have a speaking acquaintance with "Alice in Wonderland." But do you know Rev. Charles L. Dodgson, the mathematician who wrote as "Lewis Carroll?"

Did you know that he was a teacher of mathematics at Oxford, that he wrote texts in Arithmetic, Algebra, Plane and Analytic Geometry, and Trigonometry? Besides this he did a number of works on puzzles. Among them "Pillow Problems" a set of 72 problems to be worked without paper in bed.

Probably his rank as a mathematician was not high; you will note errors in these works.

As you might surmise he published a number of works on logic—among them "The Game of Logic." The barber shop paradox, which still baffles logicians can be found in this publication. Perhaps Carroll did not know how deeply he was probing into logic.

I think your students will be happy to know about "Lewis Carroll."—PHILIP PEAK, *Indiana University, Bloomington, Indiana*

Some remarks on enrichment[1]

IZAAK WIRSZUP, *Mathematic's Staff of the College, University of Chicago, Chicago, Illinois.* *Enrichment of our school mathematics curriculum can have many forms and many purposes. This paper suggests a more central role for enrichment, and reports the first conclusions of a study of the uses of enrichment in Russian schools.*

Introduction

THE REMARKS on enrichment which follow have been prompted mainly by the fact that I am one of the associate editors of *The Mathematics Student Journal.* Since one function of this journal is to enrich the student's mathematical experience, I feel some obligation to indicate what I think enrichment is, and how it might occur in the American school mathematics curriculum.

One matter should be made clear at the outset. If what I say about enrichment proves to have interest or value for you, it will be due in large measure to my membership in the Mathematics Staff of the College of the University of Chicago. For almost a decade now, this staff has given much attention to the aims, content, and pedagogy of intermediate mathematics—confronting such questions as: What are the implications of modern mathematics for the traditional mathematical subjects of high school and early college? How modernize the *content* of the mathematics curriculum of high school and early college? How modernize the *pedagogy* of intermediate mathematics? What are the *substantive* connections between mathematics and other subjects? How enlarge the text and collateral literature of intermediate mathematics? What is being done concerning these questions in other parts of the world? Any continuous conscious study of such questions obviously must ramify into ideas on enrichment. Thus what I say hereafter will inevitably reflect my membership in the Mathematics Staff of the College of the University of Chicago.

Aspects of enrichment

When I speak of enrichment, I am speaking in the first instance of enrichment for the *student.* And by such enrichment I mean *a broadening and vivifying of the student's significant mathematical experience.* In this view there are two sets of implications. The objective implications center on the notion of broadening the mathematics to which the student is directly exposed (the mathematics in his courses and in his textbooks), with reference to other significant mathematics (more exactly, mathematical principles, methods and subject matter deemed significant by the generality of present-day research mathematicians). Again, there is a set of subjective implications that center on the notion of enlivening: enrichment implies an awakening and quickening. What is to be quickened is the student's interest in mathematics. What is to be awakened is his sense of mathematical power.

There is nothing new in this general view of enrichment. And there would be little new in my remarks if I went on from here to invent more distinctions and produce a theoretical analysis of enrichment.

[1] This paper was first presented as a talk in the Enrichment Section of the 34th Annual Meeting of the National Council of Teachers of Mathematics at Milwaukee, Wisconsin, on April 13, 1956.

There would be little new and much that was dull if I were to go on to a description of our school curriculums and textbooks, and our training programs for teachers. And doubtless, there would be little new and much that was wrong if I were to connect mathematics teaching with students' values and society's needs. All these considerations bear on a full and effective treatment of enrichment, but none of them can be handled adequately without blunting my present purpose, which is: to make some *practical* suggestions about the enrichment of school mathematics.

I propose to realize this purpose in a different way. I propose to tell you briefly how enrichment is viewed and attained in certain foreign educational systems. With this information at hand, we can compare our situation with that abroad and see what enrichment practices there might be effective here.

The foreign educational systems I have in mind for this purpose are East European ones, notably the Russian and the Polish. I have chosen these systems, and in particular the Russian system, because information about them is of itself not untimely; because we know (or can readily find out) what the West Europeans do, while what is done by the Russians in school mathematics is almost unknown to American teachers; and because I am able to draw on a remarkable body of source material in Russian and Polish intermediate mathematics collected by the Chicago Mathematics Staff over a period of years.

Background to enrichment practices in East Europe

First, the curriculum: What mathematics is to be found in the years of common East European schooling that correspond to our 12-year program? In Russia and her satellites (and in Western European countries as well) all school students without exception face a fixed mathematics requirement extending through at least the last six years of common schooling. The last five of these years are devoted to courses in algebra and, parallel with these, courses in geometry and trigonometry. The material covered in these courses does not extend very much beyond that of our own *maximal* high-school programs. In Poland this five-year mathematics program is preceded by two years of parallel course work in arithmetic and in the arithmetical aspects of plane and solid geometry; the whole quite carefully done from an intuitive heuristic point of view.

These facts have some explicit consequences. No one ever becomes a physician, a sociologist, a school teacher, or an army officer, without having had these required years of mathematics. In Russia, this amounts to six years of at least six class hours a week, or about 1200 class hours in all.

Second, regarding the students: It is probably the case that the East European student is more strongly motivated towards success in school than is his opposite number here. In Russia, education—more specifically, technical education—is one of the few safe avenues to preferment and position. Further, strong motivation is necessary because students there have more work to do and must develop effective study habits earlier.

Third, the training of high-school mathematics teachers: In Russia high-school mathematics teachers receive their mathematics training almost entirely in the mathematics departments of pedagogical institutes. Before they apply for admission, matriculants at these institutes have decided to be high-school mathematics teachers, they come with a strong background in school mathematics, and they gain admission only through a competitive examination. The program of mathematical studies at these institutes occupies three or four years and includes the following matters: *within mathematics*, courses in the foundations of mathematics, in modern analysis (going well beyond calculus), in modern algebra, and in various kinds of geometry (analytic, dif-

ferential, and non-Euclidean); *about mathematics and its teaching,* several courses (this means hundreds of hours) dealing with learning problems that center on key concepts of school mathematics, with subject matter connections between school mathematics and other school subjects, with the organization and classroom procedure of school courses and with the proper exploitation of extracurricular opportunities to encourage mathematical activity on the part of the student, etc.

In connection with this last matter of mathematics teaching, the prospective teacher is made especially sensitive to psychological difficulties in the learning of school mathematics (introduction of the variable, role of axioms and proof in geometry, etc.) and is taught how to make good use of the considerable school time available to him to root these concepts in the student's mind.

The strictly mathematical part of the teacher-training program just described proceeds more slowly than the counterpart university training given there to, say, prospective mathematicians and scientists; and further, it is deliberately oriented towards school mathematics in the sense that the work in higher mathematics is presented in its relation to elementary mathematical matters. The result is a teacher who has competence respecting both the technical and the pedagogical aspects of the subject he teaches.

Some enrichment features in East European schools

There are two general sources of information on current enrichment matters in East Europe: the text and collateral literature, and the periodical literature addressed to school students and teachers of school mathematics. From these sources it is seen that enrichment material of three general sorts is made available:

I. Material of a technical and pedagogical character, designed expressly for use by the teachers and students of the pedagogical institutes, and for use by active school teachers;

II. Material of a semi-technical character, designed particularly for the use of high-school students (and incidentally, their teachers and also interested laymen) and restricted in scope to the mathematics of the high-school curriculum;

III. Material of a more technical character, designed for the joint use of high-school students and their teachers and intended to lead them out *together* from curriculum mathematics toward broader mathematical knowledge and greater mathematical power.

Let me give some examples of each sort of material, and some indications of how each is used.

Consider first the *material of type I*, that is, material designed for students and teachers of the pedagogical institutes, and for the active high school mathematics teacher. This material is of two kinds. The first kind includes books, pamphlets and papers written by the mathematics professors of the pedagogical institutes or by research mathematicians (some of them world-famous) of the universities, and intended for the mathematical education of the prospective teacher. In this category are to be found the textbooks used in the pedagogical institutes themselves: an 800-page treatment of intermediate algebra, encompassing and extending school algebra and indicating the role of modern concepts at this level; a similar treatment, in 500 pages, of trigonometry and its applications; and exhaustive treatments of geometry and of calculus in the same vein. All these books are excellent examples of the art of textbook writing. And all of them are aimed directly at training high school teachers, as may be seen in the following quotation from the preface of a calculus text for prospective teachers:

> The present text in mathematical analysis is designed principally for mathematics students in the pedagogical institutes, i.e. for prospective teachers of mathematics. This fact has determined both the *content* of the text and its *mode of exposition*. We have thought it appropriate to give significant attention to the *fundamental*

concepts of analysis, viz. function, limit, continuity. These concepts are of decisive importance in an understanding of analysis. But further, they have an independent significance because of their direct relation to school mathematics and its teaching . . .

Also in this category are collateral, or supplementary, works prepared under the direction of well-known mathematicians and sponsored by the Russian Academy of Pedagogical Sciences. The principle behind these works is to give a modern systematic exposition of the content, the basis and the ramifications of high-school mathematics. The order and scheme of presentation in such works are not those of a textbook. Rather, they aim to give a simple and complete treatment of those mathematical topics out of which the high school curriculum is constructed. Much attention is devoted to matters which, though they do not appear as such in the high-school program, are nevertheless indispensable to an adequate understanding of elementary mathematics, and which offer avenues along which the content and method of school mathematics can be extended. I know of no counterpart to these works in English. Further, I know of no counterpart in English to the teacher-training texts referred to in the preceding paragraph.

The second category of type *I* materials includes books and papers written by experienced teachers, and intended to help the practicing teacher in his classroom work. These deal in an exhaustive way with methods of instruction and effective organization of class presentation; with critiques of exposition; with the place of illustrative problems, discriminating test devices, and the like; with typical student blunders and misconceptions, and devices for correcting them; and so forth. A principal vehicle for this kind of material is the periodical *Mathematics in the School*, published *bimonthly* in 94,000 copies per issue, and edited with the greatest care.

Now let us turn to the *material of type II*, material designed for use mainly by high-school students, and restricted in scope to the content and method of school mathematics. Here again two classes of material can be distinguished. The first is not entirely unexpected: it includes problem-books with the solutions added. Teachers use these books as sources of extra exercises or of test problems. Students use them for drill or further illustration, and as an aid in preparing for competitive examinations.

The second class of type *II* material centers on what, at first glance, we might call "recreational" mathematics. The main purpose of this type of literature is to develop in younger readers a taste for mathematics and a drive to be active in mathematics. For the most part, mathematical ideas and techniques outside the school curriculum are avoided, as are the more difficult theoretical questions *in* school mathematics. This sort of literature supplies problems with an uncommon setting or an unexpected answer, interesting excursions into the history of mathematics, surprising applications of algebra or geometry in everyday life, etc. A cautionary remark is in order here. To us, recreational mathematics is mathematics that makes almost no demand on the reader's background; it can be read by those who have had school algebra, those who haven't, those who have forgotten it. The literature I am discussing now does not have this character; indeed, surprisingly little literature of the undemanding sort is published in Russia. Russian "recreational" mathematics is written under the assumption that the reader has at hand an explicit stock of information about school algebra and school geometry.

Finally, regarding *material of type III*, which is designed to lead student and teacher, together, away from the school mathematics curriculum toward broader mathematical knowledge and power. Literature of such a type is especially difficult to prepare. Hence it is a matter of some astonishment to see what fine quality this type of Russian literature actually has. To begin with, of course, there is the not-uncommon expository material that extends elementary mathematics or that

introduces topics and aspects of higher modern mathematics. Such expositions are carefully planned; each separate work has a clear purpose, and the various works supplement but do not duplicate each other. But what is uncommon in this type of literature—uncommon and of particular interest to us as teachers—is a group of books in which new mathematical theories are presented to school pupils in the following fashion: After a quite short introduction, the basic definitions are presented and illustrated; theorems are then given in the form of problems, and the book proceeds in a succession of such problem-theorems, interspersed with brief discussions, until the theory in question has been run through. This part of the book is then followed by a second part, in which the problem-theorems of the first part are solved. In this way students are introduced to new theories or sizeable fragments of new theories—including modern ones such as the theory of convex figures, the theory of numbers, the theory of map-coloring, the probability theory of random walks, and so forth.

The purpose of such presentations is not merely to inculcate the results of the new theories presented. Indeed, this aim appears to be quite secondary. Such presentations appear to be designed to foster in the student habits of independent creative work in mathematics, in deliberate contrast to the more passive assimilation of, say, the mathematics of a class text. The very material selected for treatment is such that the student's main attention must fall on the methods rather than on the results. Often a single theorem is approached in different ways, so that the student is led to make and to compare different proofs of the same result.

It is hardly necessary to repeat the obvious fact that to write books of this sort is most difficult. Usually the author has devoted years of work to preparing the material, trying it out with students in mathematics clubs and rewriting and reorganizing it several times before his presentation reaches the desired state of technical and pedagogical completeness.

The reference just made to mathematics clubs brings me to the formal instruments by which enrichment is effected with the East European student. Of course, the literature itself is the chief instrument of enrichment. But once the literature is at hand, the problem is to bring the student into an active relationship with it. Here the usual instrument is the student mathematics club. In East European schools, the authorities attach much importance to such clubs, and the students attach much prestige to membership in them. The mathematics teachers direct the clubs and devote considerable effort to planning and operating the club programs. The better students are usually the ones who lecture or direct discussion, but a free exchange is encouraged between all the members. In this way the students learn to read mathematics independently and critically, learn to discuss mathematical subjects in the language of mathematics, acquire information which supplements their classroom knowledge, and move gradually into an "inventive" relation with mathematical problems. It is to be emphasized that the clubs are not restricted to the superior students; average students are also encouraged to participate, and receive attention on the same footing as the rest.

There is an interesting variant of the school mathematics club practice just described. In Russian urban areas it is an accepted thing for the university mathematics department to assign to some of its members the task of conducting regular meetings with selected high-school students. These meetings are also devoted to enrichment material, the difference now being that the best students are put into a face-to-face relation with active mathematicians. Some university mathematics departments also sponsor public lectures by prominent mathematicians open to all high school mathematics students; frequently these lectures are published in paper-back form.

Another indirect instrument of enrich-

ment is the local association of school mathematics teachers. Teachers meet regularly to exchange experiences, to report discoveries in their teaching, and to discuss and assess recent publications bearing on the school mathematics curriculum. These meetings, of course, affect only the teacher; but they affect enrichment indirectly by keeping the teacher open and alive to possibilities in the direction of enrichment.

Some comparisons

Let us turn to some comparisons between the East European educational situation in mathematics and our own. It is well to remind ourselves at the outset that the American public school system is in many respects the finest in the world: we train the greatest number of students, our training does not warp their personalities, our educational institutions literally express the student's inalienable right to know and his inalienable right to study what he pleases, and as citizens our graduates participate creatively in the world's most intense and richly complex civilization. It is well to remind ourselves that the Russian educational system serves the state, ours the individual; that their system sees a heavy emphasis—indeed, a distortion—in favor of the technical and scientific, whereas ours moves steadily towards a general education in fundamentals; and that we, unlike the Russians, seek to *increase* the flexibility of our school programs so as to help *all* children.

It is also well to remind ourselves, at the outset, that the proper attitude towards such comparisons is a *professional* one. We teach school mathematics. So do East European teachers. How do they do it? What do they aim for? What are their students and their schools like? What are their pedagogical instruments? How successful are they? These are objective questions. They can be treated in a professional way. They should be treated by the members of the National Council, since we are *the unique group* in America with a professional responsibility for school mathematics. Should it then appear (as indeed it does appear) that East European mathematical education in the common school is more rigorous and effective than ours, this professional attitude will help us select and implement the right corrective measures respecting our own work.

Certain features of mathematics teaching in East European schools stand out at once. The subject has a thorough and complete treatment in a very closely organized curriculum. The whole sequence of mathematics courses is required of all students. Ample time is available for the teaching (note in this connection that the school year is as long as ours, but that students attend school *six* days a week). Students are strongly—if artificially—motivated toward mathematics and science; and at the same time the integration of the total school curriculum is such that mathematics learning is reinforced in the non-mathematical parts of the curriculum. Text material is of high quality, and an exceptional amount of enrichment literature is available. The mathematics teachers are unusually well trained; further, they receive systematic help with their pedagogical problems from official bodies and official publications.

There is an obvious contrast between these features and corresponding features of American mathematics training. Some particular contrasts deserve to be underscored. The upper school mathematics curriculum of East Europe is stable, and is relatively traditional in content; while the same is broadly true here, our curriculum tends to be somewhat spotty and expressions of doubt are multiplying in high quarters and low about the advisability of retaining even the traditional year of plane geometry. The mathematics topics of the East European curriculum are carefully integrated into one continuous mathematics program; the same is not true here. The East European student is taken through this program slowly, and diffi-

cult transitions (from arithmetic to algebra, from heuristic to axiomatic geometry) are made very deliberately. The work of American students is much more concentrated, is covered at a faster rate, and involves sharp discontinuities at the familiar transition points. The content of our first-year algebra is spread over two or three years in Russia, and our first-year geometry is spread over at least two years there.

These last aspects of our operations, I would add parenthetically, may go some way to account for the published estimates that only 25 per cent of our students take first-year algebra and only 11 per cent first-year geometry.[2] They may also account in part for the consensus of college teachers that high school graduates have little understanding of the mathematics they are trained in, little sense of the interdependence of mathematics and general science, small capacity for independent work in mathematics. In mathematics, the smoothing out of difficulties, the enhancement of understanding and the development of self-reliance and independence all require *time*—and time is what the American mathematics teacher lacks. Should it ever come about that we mathematics teachers are given a chance to devise and establish a different school mathematics curriculum, I hope we will be able to correct some of these disadvantageous aspects of our present one.

Turning to the particular matter of enrichment, there are further contrasts to be seen. Enrichment is deliberately practiced in East Europe; not so here. Enrichment material is plentiful there, again, not so here. Enrichment literature there has a definite technical content; what we have here tends to be non-technical and hortatory. And above all, in East Europe enrichment is viewed as enrichment for *both* the student and the teacher.

In my opinion, this last point deserves the most serious consideration. Nobody knows better than the American mathematics teacher how much he—and, to an only slightly lesser extent, the student—is left on his own in mathematics. Here is one of the great unfortunate consequences of the deep division between American mathematical research and American mathematics teaching. Today America has world pre-eminence in mathematical research. Is the American educational effort in mathematics adequate to maintain this pre-eminence?

Note, finally, how neatly our country's needs and interests in school mathematics teaching can be focussed in the area of enrichment. Contributions from specialists can take the form of enrichment materials designed for both teacher and student. The teacher's concern for the revision of the curriculum and the improvement of mathematical pedagogy can *begin* to find expression in the form of experiments with enrichment materials. In this way *enrichment can serve as the cutting edge of improvement in school mathematics and of its articulation with higher mathematics.*

Some practical suggestions

Practically speaking, to introduce enrichment practices into American school mathematics requires instruments. Chief among these of course, is the enrichment material itself. But I do not consider it sufficient simply to hand an interesting book or paper to a superior student. Some device is needed to give to enrichment a form and continuity that students can appreciate. In my view, extracurricular student mathematics clubs can best serve this purpose. My first suggestion therefore is that *a student mathematics club be organized in each high school.* Indeed, if the school is large enough I would suggest the formation of two such clubs: a *junior mathematics club,* for students who are taking first-year algebra or first-year geometry, a club for freshmen and sophomores; and a *senior mathematics club,* for students who have continued into third-year or fourth-year mathematics.

[2] See K. E. Brown, "National Enrollments in High School Mathematics," The Mathematics Teacher, XLIX (May 1956), 366.

In the junior club, the teacher would aim simply to increase (or awaken) the student's interest in mathematics. There would be no deliberate attempt to increase the student's knowledge or power; the goal would be mathematical fun. Use can be made of good mathematical riddles, novel problems solvable by the students, alternative explanations of known results, and excursions into the history of mathematics. Other topics can be found in the Council's publication *Recreational Mathematics*, published in 1955. What the student should find in all this is enjoyment, pure and simple. Since there is a good probability that students at this level will take definite satisfaction in working problems for the sheer sense of power, problem-lists should also be at hand.

In the senior club, the teacher would aim to extend somewhat the student's knowledge and mathematical power, having in mind both the next levels of mathematics education and the desirability of independence and inventiveness in mathematical work on the student's part. The student should come to share this aim with the teacher. Topics suitable for exposition in such clubs have been listed from time to time. Let me mention a few others, simply as examples: the calculus of statements; what makes a "proof" a proof; elementary descriptive set theory; sentences and their graphs; axiomatic algebra; geometrical constructions; congruences and mod-p number systems; applications of trigonometry to geometry; limits in geometry. It is also desirable to introduce some carefully-selected discussion problems, in addition to problems of the more usual type.

I regard it as important that such clubs be established as soon as possible, and probably in the order: senior club first, junior club second. But I regard it as equally important that the clubs be well established. My feeling is that at least a year of preparation should precede the formal creation of a club. During this period the teacher would rough out a reasonable program, collect materials, strengthen the school library, and do what is possible to cultivate unorganized extracurricular interest in mathematics among the students.

Admittedly, the current stock of enrichment material is low. It is reasonable to think that this stock will grow. Meanwhile, *The Mathematics Student Journal* can serve as a source of material. As issues of this journal accumulate, difficulties in building an attractive club program should decline.

These suggestions are not particularly novel, but they are practicable. They can be implemented in almost every school. If we were to put them into effect, we would take the first practical step toward the revision and revitalization of school mathematics and toward the improvement of our product, the mathematics student.

Concluding remarks

I have felt it my professional obligation to give you the information and the general views above on enrichment. But I would have been reluctant to say what I did if I could see no concrete development in this direction. However, there are some contributions I can mention and some constructive activities now afoot about which you may be interested to hear.

I remarked earlier that the College Mathematics Staff at Chicago is giving some attention to school mathematics and its enrichment. Let me now tell you something of what this staff has done and what it plans to do in this connection. You can best appreciate these activities if you see them as concrete expressions of the philosophy set forth earlier.

To begin with, the Staff has already published four articles in *The Mathematics Student Journal*, namely: "A Generalization of the Pythagorean Theorem," (Vol. 2, No. 2, April 1955); "Three Algebraic Questions Connected with Pythagoras' Theorem," (Vol. 2, No. 4, December 1955; and Vol. 3, No. 2, April 1956); "A Problem on the Cutting of Squares," (Vol. 3,

No. 2, April 1956); "More on the Cutting of Squares," (Vol. 3, No. 3, October 1956).

These four little articles are unpretentious, but they have been chosen with care to illustrate three types of enrichment material. The first is a new proof of a famous result; a proof which, incidentally, opens the way to views of other geometries. The second article produces an unexpected result, and shows how it can be reached with just the tools of elementary algebra; this article also opens the question of relating algebra and geometry. The third article deals with an entirely novel and challenging problem in geometry. The fourth gives a new and natural proof of a famous theorem of geometry. The staff of the College plans to continue this series of articles in the *Student Journal.*

I have made the point that enrichment in mathematics must be enrichment for both student and teacher. The mathematics staff at Chicago has given expression to this conviction in the following way. With the generous cooperation of the editor of *The Mathematics Student Journal* and the editor of THE MATHEMATICS TEACHER, the College staff has undertaken to publish, in THE MATHEMATICS TEACHER, articles complementary to those it publishes in *The Mathematics Student Journal.* Thus the April 1956 issue of THE MATHEMATICS TEACHER carries a paper entitled "Three Algebraic Questions Connected with Pythagoras' Theorem," whose purpose is to complement the *Student Journal* article of the same name by adding background material and giving a more general discussion of the questions treated in the *Journal* article. You will note that the TEACHER treatment appears in the same month as the second part of the *Student* article. Similar companion pieces are given for the articles on the cutting of squares. Thus, the May 1956 issue of THE MATHEMATICS TEACHER has a paper with the same title amplifying the *Student* article; teachers who work through this paper will have the pleasure of encountering a simple geometric problem whose statement is comprehensible to students but which is an unsolved mathematical problem. In the October 1956 issue of THE MATHEMATICS TEACHER there is a second paper dealing with the cutting of squares; this paper has a counterpart in the October 1956 issue of the *Student Journal.*

The staff at Chicago hopes to continue this practice of complementing *Student Journal* articles with more elaborate treatments in THE MATHEMATICS TEACHER. I should add here that we should be happy to hear from teachers on any aspect of this material or its treatment.

The College staff has the opinion that some of the enrichment material used in Russia is quite suggestive both in content and in exposition. It therefore has allotted part of its efforts to the task of making some of this material available to teachers in book or paper form. They are now at work on a little geometry book *The wonders of the square* by B. Kordiemski and N. Rusalew (a book which was published in Russia in an edition of 200,000 copies), and on the book *Convex Figures* by I. M. Yaglom and V. G. Boltianskii. Also, a program of work has been arranged for dealing with other books and periodicals, both Russian and Polish.[3] Of all this, we will be able to explain more in time.

[3] Of Russian books that can have practical value in enrichment and that are in active preparation by this Staff, I would mention here the following: Perelman, *Interesting Geometry;* Perelman, *Interesting Algebra;* Dynkin and Uspenskii, *Mathematical Conversations;* Khintchin, *Eight Lectures on Mathematical Analysis;* A. Yaglom and I. Yaglom, *Non-elementary problems in an elementary setting.*

Of Russian books that can contribute to the background of the teacher and that are in preparation: Khintchin, *A Short Course of Mathematical Analysis;* Grebentcha and Novoselov, *A Course of Mathematical Analysis,* I and II; Gniedenko and Khintchin, *Elementary Introduction to the Theory of Probability.*

On the formulation of certain arithmetical questions

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This article discusses three aspects of symbols for numbers and their relation to "simplification" of arithmetical expressions.

EVERYONE'S ANSWER to the question "$9\times4=?$" is "36." Yet "4×9," "$6+12+18$," "$100-64$," "6^2," and countless other answers would be equally correct. Why then does everyone say "36?"

Psychologically, the almost automatic response is, of course, due to the fact that when they were children the respondents memorized the multiplication tables. Similarly, because they have memorized the alphabet, they automatically answer "y" when asked which letter follows x in the alphabet. But there is more to the answer "36" than being conditioned to this response. The answer "36" is actually different from all other possible answers.

The difference arises from the fact that the symbols for integers fall into two classes: (1) *basic symbols*, in the Western world, the Hindu-Arabic numerals "1," "2," "3," and so forth; (2) *expressions*, such as "4×9" and "$6+12+18$" describing numbers by certain relations to certain other numbers: as the product of 4 and 9, the sum of 6 and 12 and 18, the results of multiplications, additions, and the like. Expressions describe numbers in terms of basic numerals and symbols for operations (such as "$\times$" for multiplication, and "$+$" for addition).

The first point of this paper concerns the clarification of the nature of countless arithmetical questions including, besides $9\times4=?$, problems with the preambles "find the product . . . ," "compute the sum . . . ," "determine the quotient . . . ," and the like. *Each such question aims at the basic symbol for a number described by an expression.*

It would seem to be highly desirable that students of arithmetic be acquainted with this situation at an early stage of their training. Students do not have to be very mature to grasp the difference between basic symbols and expressions. And only if the nature of those arithmetical questions is understood are the problems raised above the level of questions such as "Which letter follows x in the alphabet?"

For rational numbers, there are two commonly used sets of basic numerals: decimal fraction symbols,[1] such as .5, .6857, .066 . . . 6 . . . or $.\dot{0}\dot{6}$, and $.\dot{1}\dot{4}\dot{2}\dot{8}\dot{5}\dot{7}$; and common fraction symbols such as 1/2, 11/16, 2/30, and 1/7. As these examples show, in some cases the decimal fraction symbol is shorter, in others the common fraction symbol is shorter.

Each type of symbol conveys certain information about the numbers better than does the other. For instance, a glance at $.\dot{6}\dot{2}\dot{1}$ and $.\dot{6}\dot{4}\dot{1}\dot{0}\dot{2}\dot{5}$ tells one that the first number is smaller than the second. It takes most people longer to deduce this fact from the common fraction symbols for these numbers, 23/37 and 25/39. To compare even such relatively simple fractions

[1] We write .5 and ½ instead of ".5" and "½" where the words "the symbol" make clear that reference is made to the numeral and not to the number.

they often replace them by the corresponding decimal fraction symbols. On the other hand, for the product of the two numbers mentioned, it certainly is easier to find the common fraction symbol from the common fractions than the decimal fraction symbol from the decimal fractions. As an extreme example, contrast the ease with which one learns to recognize the product of 7 and 1/7 as 1 with the difficulty in recognizing the product of 7 and $.\dot{1}42857\dot{}$. The greater fitness of common fraction symbols in multiplications and divisions is not surprising since they are defined in terms of multiplication alone, whereas decimal fraction symbols are defined in terms of multiplication and addition.

Which of the two systems one chooses in a given case will depend largely upon the nature and the purpose of the study. For instance, in printing comparative information, especially tables, decimal fraction symbols are preferred.

The second point of this paper, illustrated by the preceding examples, is to stress *the existence of various possible systems of basic numerals.* Again it seems desirable that students of arithmetic be familiarized with the situation. Each question aiming at the basic symbol for a number given by an expression obviously presupposes a clear specification of a definite system of basic symbols.

On the level of instruction where non-decimal positional symbols are introduced, dual and duodecimal numerals (using the bases two and twelve, respectively) supply further examples.

Among the irrational numbers are some whose (necessarily non-periodic) decimal fraction representations can be described by simple rules. For instance, .101001000100001 . . . and so on can be described by saying that all its digits are zeros except the $\frac{1}{2}n(n+1)-st$ digit which, for any n, is one.

On the other hand, for the positive number whose square is 2, there is no simple rule describing the entire string of digits although one may compute as many digits as he pleases. True, when one reads "1.41412 . . . " he imagines that a reference to the number in question is intended. But this is a mere guess, since countless other numbers begin with those same five decimals and since the symbol does not in any way indicate what digits follow (except possibly that an unending string is to be expected). The only complete and unambiguous description of the positive number whose square is 2 is by an expression such as $\sqrt{2}$, consisting of one or more symbols for relations or operations and of one or more basic numerals.

The third and perhaps most important point of this paper is to stress that, *for many numbers, expressions are not only the simplest symbols but the only available symbols that are complete and unambiguous.*

This insight seems to have an important bearing on questions aiming at the basic symbols for numbers. Current texts in arithmetic and algebra abound in problems such as: Compute $\sqrt{16}-\sqrt{9}$. Perform the indicated operations: $\sqrt[3]{8}+\sqrt[3]{27}$.

In such problems expressions designating numbers for which simple basic numerals are available have been selected. But very similar expressions designate numbers for which simple basic numerals do not exist, e.g., $\sqrt[3]{7}+\sqrt[3]{26}$. It would, of course, be perfectly reasonable to ask: Find the first five digits of $\sqrt[3]{7}+\sqrt[3]{26}$. But it would not be reasonable to ask: Perform the operations indicated in $\sqrt[3]{7}+\sqrt[3]{26}$. What could one do but replace the complete and unambiguous expression by one of countless incomplete symbols none of which characterizes the number designated? On the other hand, the difference between the problems involving $\sqrt[3]{7}$ and $\sqrt[3]{8}$ does not concern the relations or operations. It lies in the fact that the expressions designate numbers one of which does, and one of which does not, have an available decimal fraction symbol.

The student of arithmetic and algebra should not only be trained in finding the basic numerals for some numbers de-

scribed by expressions, but also in understanding that, for many numbers, expressions are the simplest, or even the only complete and unambiguous descriptions, and that in such cases replacement of the expression by a basic decimal numeral must be foregone.

We conclude with an example of what we believe is a reasonable and desirable formulation of problems of the type mentioned previously.

For each of the following numbers try to find the basic decimal numeral, or, if the latter is not available, a shorter expression (say, one that includes fewer root signs). Note that in some cases even such a "simplification" is impossible: a) $\sqrt{4.41}+\sqrt{4.84}+\sqrt{5.29}$; b) $3\sqrt{5}$; c) $\sqrt{3}-\sqrt{2}$; d) $\log_{10}\sqrt{10}$; e) $\sqrt{2+\log_{10}100}$; g) $\sqrt{2}\cos 45°$; h) $1+\cos 45°$.

Similarly, consider the sum, the difference, the product, and the quotient of the following pairs of numbers: i) $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$; j) $\sqrt[3]{3}+\sqrt[3]{2}$ and $\sqrt[3]{3}-\sqrt[3]{2}$; k) $\sqrt[3]{3}+\sqrt[3]{2}$ and $\sqrt[3]{7}-\sqrt[3]{5}$.

"Of his many educations, Adams thought that of the school teacher the thinnest. Yet he was forced to admit that the education of an editor was thinner still."—*Henry Adams (1838–1918), in* The Education of Henry Adams.

A DATE TO REMEMBER

Seventeenth Christmas Meeting
NCTM

December 26–29, 1956
Arkansas State College, Jonesboro Arkansas

An introduction to negative integers

CHARLES BRUMFIEL, *Ball State Teachers College, Muncie, Indiana.*

A new algebra for the secondary schools will require new methods.

THIS ARTICLE describes a verbal presentation of the concept of negative numbers to a ten-year-old boy of better than average intelligence. The boy had been encouraged to develop his ability to think logically. Note that the youngster derived for himself the rules usually stated in textbooks without much rationalization:

1. When adding two numbers of like sign, add the absolute values and prefix the common sign.

2. When adding two numbers of unlike sign, subtract the lesser of the absolute values from the greater and prefix the sign of the number of greater absolute value.

The following simple discussion gave the boy an understanding that led with little computation to a mastery of addition and subtraction of integers. A classroom presentation along the same lines seems feasible. The *absence* of written symbols probably aided the learning process.

We submit the conversation between the teacher, "I," and the boy, "G."

I: What number added to seven makes twelve?

G: Five.

I: What number added to four makes zero?

G: There isn't any such number.

I: Well, no number that you know will do. But, now I am going to create a new number for you that will do the job. We shall call it *negative four,* and when we add four to *negative four* we do get zero. Now, how about a number that will give zero when added to five?

G: I suppose we'll call it *negative five.*

I: And what is the number *negative eight?*

G: The number which added to eight makes zero.

I: And what number added to one million makes zero?

G: *Negative million.*

I: Good. Now you understand what *negative number* means. Can you add six and *negative four?*

G: I'll break the six up into four and two. The four and *negative four* make zero; so the answer is two.

I: How about twelve and *negative nine?*

G: Three.

I: Can you add *negative four* to *negative five?*

G (*after some hesitation*): I think the answer is *negative nine.*

I: Prove it to me.

G (*more hesitation*): I guess I can't.

I: What is the number *negative nine?*

G: The number which gives zero when you add it to nine.

I: Then how can you find out whether *negative four* and *negative five* make *negative nine?*

G (*slight hesitation*): Oh, I see. In order to show that *negative five* and *negative four* make *negative nine,* all I have to do is show that when I add them to nine I do get zero. Well, I just break nine up into four and five. The *negative four* and four make zero. The *negative five* and five are zero. Zero plus zero is zero. So I do get zero. And *negative four* plus *negative five* is *negative nine.*

I: Fine. And now, how about adding seven and *negative ten?*

G: That's easy. I just break the *negative ten* up into *negative seven* and *negative three,* and I get *negative three.*

I: And how do you know you can break the *negative ten* up so?

G: We just did that. It is like the *negative four* and *negative five* making *negative nine*.

This completed the discussion. Some time later, subtraction was taught after a brief review of the foregoing discussion. The definition of subtraction was considered, namely: To subtract a from b is to determine a unique number c such that $a+c=b$. Subtraction of integers was then almost immediately mastered.

The reader will note the important role played by the key question: What is a negative number? Also, observe the absence of the term *positive*. The order of the presentation is worth noting. Using the symbols a and b to represent natural numbers, that is, numbers belonging to the set, $\{1, 2, 3, \cdots\}$, we considered in turn problems of the form:

$$(1)\ a+(-b); \quad a>b$$

$$(2)\ (-a)+(-b)$$

$$(3)\ a+(-b); \quad a<b$$

Undoubtedly the verbal presentation was more easily grasped than one which would have combined verbal and written symbols. At a later time, a notation for negative numbers was introduced. The use of the symbol "$-$" to designate a negative number, and its association with the operation of subtraction, probably makes it more difficult to understand operations with integers.

As an interesting application of the same teaching technique, some months later the following discussion occurred:

I: Pretend that you never heard of zero and only know about the numbers one, two, three, etc.

G: All right.

I: What number added to one gives one?

G: There isn't any.

I: I'll invent one. We'll call it *zero* and define it so: one plus zero is one. Now, add two and zero.

G: Well, *all I know is that one and zero make one*, so I'll break the two into one and one. The first one and zero make one; one and one is two. So, two and zero are two.

I: And how about five and zero?

G: I break five into four and one. The answer is five.

I: How about n and zero?

G: I'll split n into n minus one and one. One and zero are one. One and n minus one make n.

I: Now what do we know?

G: Zero added to any number just gives you the same number.

I: Now let us learn how to use negative numbers in multiplication.

G: All right.

I: How do you suppose we shall define four times *negative five*?

G: I suppose it means to add four negative five's together. That would be *negative twenty*.

I: That is indeed what it means. What does one times *negative five* mean?

G: *That doesn't mean anything*. You can't add one *negative five*. But, I suppose we call it *negative five*, just as we say one times four is four.

I: That is right. Now how much is six times *negative eight*?

G: *Negative forty-eight.*

I: What about *negative two* times five?

G: *That doesn't mean anything. You can't add negative two fives.*

I: Of course not. But we make a new agreement. You know that three times four is the same as four times three. We do the same sort of thing with *negative two* times five.

G: Oh, I see. *Negative two* times five means five times *negative two*. That's *negative ten*.

I: What shall we do with *negative two* times *negative three*?

G: Let's see. It can't be *negative six* because two times *negative three* is *negative six*. It must be six. But, I don't know . . . *It really doesn't mean anything*, because you can't add *negative two negative threes*

and you can't add *negative three negative twos.*

I: We do agree that it is six, and I'll show you why. We'll have to write some things down in order to see the reason for our agreement.

(*The notation "-3" for negative three, etc., have been introduced already.*)

I: You know that (*writing*) $4(5+3) = 4\cdot 8 = 32$ or, computing it differently: $4(5+3) = (4\cdot 5)+(4\cdot 3) = 20+12 = 32$.

G: Yes. You can add first and then multiply, or multiply first and then add. You get the same answer.

I: Now, when we use negative numbers in multiplication we certainly want to be able to do the same sort of thing For example, let me show you another reason for calling three times *negative four* a *negative twelve.* What is (*writing*) $3[4+(-4)]$?

G: Four and *negative four* are zero and three times zero makes zero.

I: Right. Now let us multiply first and then add. Pretend you don't know what three times *negative four* is (*writing*)

$$3[4+(-4)] = (3\cdot 4)+[3\cdot(-4)] = ?$$

What do we want our answer to be?

G: Zero.

I: Then what do we have to call three times *negative four?*

G: I see. Three times four is twelve and three times *negative four* has to be something which added to twelve gives zero. So, three times *negative four* has to be *negative twelve.*

I: Now let's go back to our original question. What are we going to call *negative two* times *negative three*? We look at

$$(-2)[3+(-3)] = ?$$

What is the answer?

G: Three plus *negative three* is zero and *negative two* times zero must be zero, so the answer is zero.

I: That's all right. But, how can you be sure that *negative two* times zero is zero? Have we ever met a product like that?

G: No, we haven't, but zero times every thing else is zero.

I: Well, let's see why we call *negative two* times zero, zero. We write

$$(-2)[(4+0)] = ?$$

G: *Negative eight.*

I: Now, multiply out first:

$$(-2)[(4+0)] = (-8)+(-2)\cdot 0 = ?$$

G: I see. We have to call it zero in order to get *negative eight.*

I: Now, back to our *negative two* and *negative three*. We are sure that

$$(-2)[3+(-3)] = 0.$$

Let's multiply first and then add. What do we get?

G: Well, *negative two* times three is *negative six*. Then we have *negative two* times *negative three* to add on to the *negative six*. But, we know the answer is zero. Oh, I see. The only number you can add to *negative six* to get zero is six. So, we have to call *negative two* times *negative three* six.

I: That's the idea. *Nothing else would work* if we want our old rules to hold. Now you use the same method to explain to me why we call *negative four* times *negative five* twenty.

The boy wrote,

$$(-4)[5+(-5)] = 0$$

and presented substantially the same argument as above. Since he had been taught some of the symbolism of algebra, the lesson continued. Using letters to represent numbers, it was pointed out that since

$$a[b+(-b)] = 0,$$

and we surely want to have the same number when we multiply first

$$a[b+(-b)] = ab+a(-b);$$

then we must call a times *negative b negative ab*. Similarly, since

$$(-a)[b+(-b)] = 0,$$

and we desire, multiplying first,

$$(-a)[b+(-b)] = -(ab)+(-a)(-b),$$

to have the same value, we are forced to say the product of *negative a* and *negative b* must be ab. These ideas seemed to be clearly grasped, and after an explanation "G" was able to duplicate the foregoing symbolic notation and make proper verbal statements explaining the necessity for having

$$a(-b) = -(ab) \quad \text{and} \quad (-a)(-b) = ab.$$

We submit the following opinions:

1. Logical thinking can be taught on every level.

2. It pays to teach correct habits of logical thinking.

3. Many times an abstract approach to a number concept leads to understanding more quickly than a manipulative approach.

4. When new concepts are introduced, clear descriptions in terms of old familiar concepts should be formulated. These descriptions should be continually referred to as necessity arises during the reasoning process that explores applications of the new concepts.

Mathematics teaching essay contest

Kappa Mu Epsilon and the Science Teaching Improvement Program (STIP) of the American Association for the Advancement of Science (AAAS) are cooperating in the sponsorship of an essay contest on the subject, "Opportunities in Teaching Mathematics in Secondary Schools." Satisfactory essays will be published in *The Pentagon*. The prize-winning essays will also be offered for publication in *The Mathematics Student Journal*.

Purpose of the contest

The Mathematics Teaching Essay Contest is planned to increase interest in the teaching of mathematics at the secondary-school level. It is intended to encourage undergraduate and graduate students in mathematics to consider the advantages of a career in secondary-school mathematics teaching. It is also hoped that the preparation, as well as the reading, of the essays, may interest good students with an aptitude for, and interest in, mathematics to enter the teaching profession. The importance of the ability to express oneself in writing, particularly on the part of teachers, should also be emphasized by such an essay contest.

First prize in the contest will be \$50. There will be second and third prizes of \$25 and \$15, respectively.

Conditions of the contest

1. Essays submitted in the contest should reach Professor Carl B. Fronabarger, Southwest Missouri State College, Springfield, Missouri, no later than April 1, 1957.

2. The essays must not be more than 1,000 words in length, and should be typed, double-spaced, on a good grade of paper. Four (4) copies should be submitted by each contestant.

3. The content of the essay should be as specific as possible, and should be intended to point out the advantages of preparation for the teaching of mathematics at the secondary-school level. The essay may consider one or more of the special facets of the profession of mathematics teaching, or cover the general area as completely as the length of the essay will permit.

4. Undergraduate and graduate students in mathematics are eligible to enter the contest.

5. Essays submitted will become the property of Kappa Mu Epsilon and STIP.

6. The essays will be judged on accuracy and objectivity of the data presented, the degree to which the essay appears to be convincing in the case presented for mathematics teaching, and composition and neatness of the essay.

7. The judges of the contest will be a panel of five (5) college and secondary-school teachers of mathematics.

8. A bibliography of source material should be included.

The status of the secondary mathematics program for the talented

R. A. BAUMGARTNER, *Freeport Senior High School, Freeport, Illinois.*

There is an increasing interest in revising the mathematics program to take care more adequately of the talented high school pupil.

THERE HAS BEEN a great interest in the last few years in the education of the superior and gifted child.[1] This interest, however, seems to have had little effect in the field of mathematics where enrollments in grades 10 through 12 have fallen off 16 per cent from 1934 to 1952.[2] We are all aware of the shortage of scientists and engineers. By 1960, it appears this shortage will be critical.[3]

If the supply of scientists, engineers, and technicians is to be increased, more boys and girls need additional and more challenging experiences in mathematics. In this connection, the Report of the Committee on Human Resources states, "Society fails to secure the full benefit of many of its highest youth because they do not secure the education that would enable them to work at the levels for which they are potentially qualified."[4]

Who, then, are the boys and girls who should take additional mathematics of a challenging nature? How can we identify them? Fehr[5] points out that traits of the superior in mathematics include an extraordinary memory, high-level abstract thinking, application of knowledge, intellectual curiosity, and facility of expression. These traits refer to the superior in many fields. Intelligence, past mathematics records, and teachers' estimates are also important in identifying the superior. According to the pamphlet, *Education for the Talented in Mathematics and Science*,[6] pupils who show an interest in science and mathematics, and who are in the upper 20 per cent in intelligence, may be classified as talented in mathematics and science. However, many more pupils than these can profit from mathematics courses in grades 10 and 11, and perhaps even in grade 12. Mathematics teachers can do much to help guide boys and girls into advanced mathematics classes by observing them in their present classes. In some large high schools, the mathematics department keeps records of pupils' accomplishments. These are used in helping boys and girls plan their future mathematics courses.

Guidance and recognition of the talented will not be altogether successful in securing more pupils for advanced mathematics unless we have curriculum materials and types of teaching which meet the needs and interests of our boys and girls. The following, then, is an outline of some state

[1] *Education of the Gifted* (Washington, D.C.: National Education Association, 1950.)

[2] Kenneth E. Brown, *Mathematics in Public High Schools*, U.S. Office of Education, Bulletin, 1953, No. 5, p. 33.

[3] Dael Wolfe, "Future Supply of Science and Mathematics Students," THE MATHEMATICS TEACHER, XLVI (May, 1953), pp. 227–29.

[4] *America's Resources of Specialized Talent*, Report of Human Resources and Advanced Training (New York: Harper & Bros., 1954), p. 269.

[5] Howard F. Fehr, "Mathematics for the Gifted," *Bulletin of the National Association of Secondary School Principals*, XXXVIII (May, 1954), pp. 103–10.

[6] Kenneth Brown and Philip G. Johnson, *Education for the Talented in Mathematics and Science*, U.S. Office of Education, Bulletin, 1952, no. 5, p. 2.

curriculum materials, certain school materials, and articles that have been recently published. Much of this will be particularly useful to larger schools.

New York has curriculum materials of interest to all.[7] In the tenth grade, geometry is still basic although the number of theorems has been reduced. However, the geometry is integrated with arithmetic, algebra, and numerical trigonometry. Grade 11 is comprised of an integrated course of intermediate algebra, plane trigonometry, and co-ordinate geometry. Twelfth-year mathematics is divided into two parts: the first semester continues the work in algebra and co-ordinate geometry with an informal introduction to the theory of limits which includes some differentiation. The second semester includes a study of lines and planes in space, three dimensional loci, mensuration of solids with emphasis on logarithmic computation, and spherical geometry.

The mathematics program at the University High School of the State University of Iowa follows the Iowa state outline.[8] In grade 9 general mathematics is offered (about a semester of algebra is included in this course). Grade 10 includes intermediate algebra while grade 11 includes plane and solid geometry. The developmental approach rather than memorization is stressed. The twelfth year of mathematics consists of a fourth semester of algebra and trigonometry. At the University High School, the subject matter of the last year is integrated. The concept of limit and its applications to problems of mathematics in the form of the derivative and integral, along with techniques of calculating these limits, are considered.

The State of Minnesota[9] has for its tenth-grade course plane geometry. Eleventh-grade mathematics includes intermediate algebra, simple trigonometry, and information on the conic sections. Grade 12 includes solid geometry and trigonometry. A unit on spherical trigonometry is included in solid geometry.

Grades 9 and 10 are primarily concerned with algebra in the Pennsylvania[10] course of study. These two years take the student through quadratic equations. Grade 11 deals mainly with plane and solid geometry and a short unit on informal trigonometry. The teaching of geometry is through the developmental approach. Twelfth year includes trigonometry (one semester), advanced algebra, statistics, solid geometry, and some analytical geometry.

A four-year course of functional mathematics has been developed in Florida.[11] Pupils may take the traditional courses, algebra in grades 9 and 10, geometry in grade 11, and trigonometry and solid geometry in grade 12.

Phillips Exeter Academy,[12] Exeter, New Hampshire, sections students according to ability with a more thorough course required of the best students. Ninth grade is primarily algebra, although some time is spent on geometry. Tenth and eleventh years are spent on a co-ordinated course of algebra and geometry with some trigonometry. Twelfth year is a combination of trigonometry, solid geometry, and a four-week unit of theory of equations. At the end of the ninth grade, 12 to 15 of the best students are selected for a special class in mathematics. These pupils take the equivalent of tenth and eleventh year mathematics in one year. In grade 12, these pupils take a combination course of analytic geometry and calculus.

[7] *Mathematics 10–11–12. An Integrated Sequence for the Senior High School Grades.* Bureau of Secondary Curriculum Development, State Education Department, Albany, N.Y., 1954.

[8] *Mathematics Series, Senior High School.* Iowa Secondary School Program XIX, State Department of Public Instruction, Des Moines, 1949.

[9] *A Guide for Instruction in Mathematics. Secondary School 7–12.* Curriculum Bulletin No. 20, Department of Education, St. Paul, 1953.

[10] *A Course of Study for Secondary Schools.* Bulletin No. 360, Department of Public Instruction, Harrisburg, Pa., 1952.

[11] *Functional Mathematics in the Secondary Schools.* Bulletin No. 36, State Department of Education, Thomas D. Bailey, Supt., Tallahassee, Fla., 1950.

[12] Data furnished by letter from Jackson B. Adkins, instructor in mathematics, Phillips Exeter Academy, Exeter, N.H.

Lyons Township High School,[13] LaGrange, Illinois, offers a twelfth-grade program in trigonometry and college algebra. The analytical aspects of trigonometry are stressed. The solution of the right spherical triangle is included. In college algebra, topics covered include theory of equations, determinants, probability, and infinite series. Stress is placed on understanding the theory. Some effort is made to explain the meaning of derivative.

The Bronx High School of Science,[14] of New York City, has four and one-half years of mathematics. This includes ninth-grade algebra and tenth-grade geometry. These are followed by a half-year-course each of intermediate algebra, trigonometry, solid geometry, advanced algebra, and higher general mathematics. The course in higher general mathematics includes such topics as: the development of the number systems, determinants, an introduction to differential calculus, curve tracing, statistics, and empirical equations. In addition, the Bronx School of Science carries on an extensive extra-curricular program. There are interscholastic competitive examinations in the New York area sponsored by the Mathematical Association of America and Pi Mu Epsilon. Pupils on these teams have opportunities for problem solving in mathematics, carrying on individual research, and developing individual projects.

Seattle Public Schools[15] teach a twelfth-year course in mathematical analysis including some trigonometry, college algebra, analytic geometry, and calculus.

The University of Illinois has approved a number of schools to give a fourth year of mathematics in trigonometry and college algebra.[16] Pupils taking these courses and doing satisfactory work in them may receive five hours of college credit providing they are not used for admission to the University of Illinois. This plan has been in operation at Freeport High School for over 25 years. Freeport graduates who enter the University of Illinois begin their college mathematics with analytic geometry and complete calculus in an additional two semesters.

Eleventh-year mathematics in many Illinois schools consists of a semester of algebra followed by a semester of solid geometry. Many pupils not interested in solid geometry the second semester discontinue their mathematical training at the end of the first semester. Some schools, including Freeport High School, have two distinct courses the second semester, one a continuation of algebra, and the other solid geometry. Included in the eleventh year, second semester algebra course is some trigonometry, analytic geometry, statistics, and introduction to the idea of limit. With a background of this type, an integrated course of college algebra, trigonometry, analytic geometry, and introduction to calculus is possible the twelfth year.

The University of Illinois Committee on Secondary Mathematics[17] has developed a new course for the ninth grade. Its content includes the number line, equations in one general number, the co-ordinate plane, angles, polygons, circles, similar figures, principles of measurement, and indirect methods of measurement. Other courses are now in preparation for grades 10, 11, and 12. These courses will prepare the student to enter the University of Illinois College of Engineering. In fact, there is a strong possibility that only three years of this program will prepare the more capable student for a college course in analytic geometry

[13] Data furnished by letter from Frank Allen, instructor of Mathematics, Lyons Township High School, LaGrange, Ill.

[14] Data received by letter from the Mathematics Department, Bronx High School of Science, New York City.

[15] Data furnished by letter from Elizabeth Roudebush, director of Mathematics, Seattle Public Schools, Seattle, Wash.

[16] *Mathematical Needs of Prospective Students at the College of Engineering of the University of Illinois.* University of Illinois Bulletin, Urbana, 1951.

[17] *High School Mathematics—First Course* (Urbana: University of Illinois, 1954).

The program of mathematics at the Wisconsin High School, University of Wisconsin,[18] consists of algebra in grade 9. Tenth year includes plane geometry with topics from solid geometry, trigonometry, and a unit on co-ordinate geometry. Grade 11 is composed of trigonometry and intermediate algebra. Grade 12 is called elementary mathematical analysis. This includes topics from college algebra, trigonometry, analytic geometry, and introduction to the calculus. Pupils entering the University of Wisconsin can complete the ordinary four semesters of work through calculus in three semesters.

The work done by the School and College Study of Admission with Advanced Standing[19] is of interest. Under the leadership of President Gordon Chalmers, of Kenyon College, this study was undertaken by 12 colleges and 27 secondary schools to see if it would be possible to admit pupils to college with advanced standing to be determined by examination. It was proposed that an agreement should be reached on the content of each subject in the secondary school to be considered in this study, one of which was mathematics. The committee decided that the twelfth-year mathematics course should be substantially calculus and analytic geometry. This meant a reorganization of the mathematics curriculum in grades 10, 11, and 12.

In the tenth grade, materials from algebra as well as geometry are used. Solid geometry and analytic geometry are included. Equations of simple curves (circle, parabola) are introduced. The eleventh year is an introduction to elementary analysis. Subject matter includes algebra, analytic geometry, trigonometry, theory of equations, remainder theorem, determinants, complex numbers, logarithms, and analytic geometry. The study of trigonometry is largely analytical in nature. The twelfth-year program gives an introduction to differential and integral calculus. Skills in trigonometry, algebra, and analytic geometry are further developed as they are employed in this course. Under this plan,[20] students may apply for advanced standing in certain colleges by means of appropriate examination. This may include college credit.

The committee report, *General Education in School and College*,[21] made by members of the faculties of several eastern schools, points out the mathematics curriculum is extremely crowded Once mathematical ideas are understood, it is not necessary to go over them many times. By eliminating solid geometry, which has only one important idea—that of spatial relationship—it would be possible to include the ideas of calculus and statistics. The report goes on to point out that the basic idea of the twelfth year should be the introduction of calculus.

Reeve[22] presents a complete general mathematics program from grades 7 to 12. Some algebra and informal geometry are introduced in grades 7 and 8 making it possible to include algebra, numerical trigonometry, arithmetic, and introduction to demonstrative geometry in grade 9. In the tenth grade, demonstrative geometry would be the central topic. Solid geometry should be included in grade 10. Numerical trigonometry in the tenth grade

[18] John R. Mayor, "A Suggested Course of Study in Mathematics," *Bulletin of the National Association of Secondary School Principals*, XXXVIII (May, 1954), pp. 80–87.

[19] H. W. Brinkman, "Mathematics in the Secondary Schools for the Exceptional Student," *The American Mathematical Monthly*, LIV (May, 1954), pp. 319–23.

[20] William H. Cornog, "The High School Can Educate the Exceptionally Able Student," *Bulletin of the National Association of Secondary School Principals*, LXXXIX (April, 1955), pp. 380–86.

[21] *General Education in School and College.* A committee report by members of the faculties of Andover, Exeter, Lawrenceville, Harvard, Princeton, and Yale. (Cambridge: Harvard University Press, 1952), pp. 52–57.

[22] W. D. Reeve, *Mathematics for the Secondary School—Its Content and Method of Teaching and Learning* (New York: Henry Holt & Co., 1954), chap. 11. See also: W. D. Reeve, "The Need for a National Policy and Program in Secondary Mathematics," The Mathematics Teacher, XLVIII (January, 1955) pp. 2–9.

includes solution of right triangles, simple identities, and introduction to the laws of sines and cosines. The eleventh grade is an integrated course in trigonometry, introduction to analytic geometry, and calculus. Grade 12 includes college algebra, analytic geometry, and the essential ideas of calculus. With gradual introduction of topics at an earlier level, it is possible to eliminate the long reviews and introduce new materials.

Sometimes by looking at industry in the local community, mathematical needs for the high school graduate planning to go into industry can be found. Recently at a sectional meeting of the Illinois Council of Teachers of Mathematics, Dr. J. J. Coleman, chief engineer, Burgess Battery Company, Freeport, Illinois, made this statement: "The practical engineer (high school graduate) must be able to read technical literature. He becomes competent in an area—perhaps limited—by self-education; and unless his mathematical training is wider and deeper than it was in the past, he will find even elementary technical literature written in a foreign language. He meets the words velocity, acceleration, electric current, magnetic flux, momentum, slope, tangent—scores of words that mean derivative. The notion of plotting a changing variable against time and the representation of rate of change as a tangent will carry him a long way."

Many teachers have pupils write term papers or complete projects in mathematics classes. Projects must be of a high mathematical nature in order that boys and girls are led to scientific research. Topics in modern mathematics[23] might be of interest to high school pupils for their projects.

Acceleration of highly gifted high school students has been tried at the University of Chicago and more recently at other colleges under the leadership of the Ford Foundation.[24]

Small high schools have a great problem in challenging the superior student in mathematics. Alternating of courses, projects, and correspondence courses offer opportunities for enrichment.[25]

Finally, the teachers of mathematics should not be forgotten.[26] The responsibility of the mathematics teacher in this highly technological age is great. We must, then, as teachers of the talented in mathematics, offer constant stimulation to the reasoning and the imagination of our pupils.

In summary, it is difficult to make any definite conclusions, on the basis of such a limited survey. Space will not permit the inclusion of many other excellent mathematics programs.

Fehr[27] has pointed out these three trends: (1) the elimination of solid geometry as a half-year subject, (2) the condensation of treatments of complex numbers, logarithmic solutions of triangles, and geometry of the circle, and (3) the addition of statistics and elementary notions of calculus.

In addition, the following points seem to be evident:

1. Unless economic and technological conditions change drastically, it is imperative for more boys and girls to take mathematics in high school in order to meet the need for scientists, engineers, and practical engineers. At present only 10 per cent of the twelfth-grade students take mathematics.[28]

2. There seems to be a definite trend to include some calculus in the high school. Courses of study and schools vary in the

[23] Saunders MacLane, "The Impact of Modern Mathematics," *Bulletin of the National Association of Secondary School Principals*, XXXVIII (May, 1954), pp. 66–70.

[24] Morris Meister, "The Ford Foundation Experiments and Their Implication for the Science Education of High-Ability Youth," *The Science Teacher*, XX (April, 1953), pp. 107–10.

[25] Mayor, *op. cit.* p. 82.

[26] Donovan A. Johnson, "Let's Do Something for the Gifted in Mathematics," The Mathematics Teacher, XLVI (May, 1953), pp. 322–25.

[27] Fehr, *op. cit.*, p. 107.

[28] *Mathematics in Public High Schools*, *op. cit.* p. 40.

amount of calculus to be offered in the twelfth year. Three state courses of study examined do not mention calculus in the twelfth year—Iowa, Minnesota, and Florida. However, the University High School at the State University of Iowa does include some calculus in the twelfth year.

3. There seems to be a decreasing emphasis on solid geometry. However, New York State, Florida, Minnesota, and the Bronx School of Science have a semester course in solid geometry. This may be due to engineering-school entrance requirements. The content of solid geometry has broadened to include trigonometry and analytical geometry of three dimensions.

4. While subject titles have not changed, state courses of study and schools are including parts of other courses in these subjects. Co-ordinate geometry is a part of plane geometry. Numerical trigonometry is found in algebra. There is no definite pattern.

5. There seems to be some need, as Reeve[29] has pointed out, for a national policy determined by high school and college teachers on co-ordination and grade placement of the subject matter of mathematics in high school. Perhaps a series of workshops involving many persons, including teachers and experts, could eventually lead to a program, organized on the basis of units, widely accepted and used. Such a plan in New York State was completed by using the ideas and suggestions of 1000 school people through a series of workshops.[30]

[29] Reeve, *op. cit.*

[30] *Mathematics for All High School Youth*, Bureau of Secondary Curriculum Development, The State Education Department, Albany, N.Y., 1953.

Manpower facts for teachers

The number of college graduates qualified to teach science in the high schools has declined annually since 1950.

Little more than half of the 4,000 graduates who qualified to teach science in 1954 entered the teaching field.

About 46 per cent of the high-school graduates with IQ of 130 or above go to college, and only about 1.7 per cent of these take Ph.D. degrees.

The Soviet Union has turned out almost twice as many technical specialists as the United States during the past 25 years.

There are nearly six million college graduates in the United States and two million in the Soviet Union. But the number of Soviet applied scientists with higher education is already equal to or greater than the number in the United States.—*Taken from the* Scientific American, *January 1956.*

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Irrationals or incommensurables V: their admission to the realm of numbers

We have seen in earlier notes[1] that incommensurable quantities were first discovered in geometric situations and were proved to exist by using the theory of evens and odds. As roots of equations, they were represented by symbols with which operations were performed, and they were approximated by rational numbers. Viete's and Descartes' literal symbolism helped to free these numbers from what in one sense was a too close association with geometric and physical magnitude. On the other hand Fermat's and Descartes' analytic geometry produced a need to associate a number with every point of a line—a feat impossible without the recognition of irrational numbers as abstract entities on a par with other numbers. This was a concept significantly different from the Greek idea of incommensurable magnitudes for whose ratios Eudoxus derived a treatment.

We might note parenthetically, however, that one French geometer, Legendre (1752–1833), in writing an elementary geometry to replace Euclid as a school text, proved propositions on similitude by applying algebraic-numerical reasoning to literal symbols which represented lengths. He thus turned the tables on the Greeks, who used lengths to represent numbers. But Legendre did not give a clear or modern exposition of irrationals.

A second motivation which we can now see would drive mathematicians to seek to clarify the idea of the irrational was the development of the calculus with its limits and problems of continuity. Augustin Cauchy, one of the prime movers in the rigorization of the calculus, in his *Cours d'Analyse* of 1821 treated irrationals as essential quantities which were familiar to all. He remarked that an irrational number could be the limit of a sequence of rationals. Although this is essentially the idea used later by Cantor, who *defined* irrationals as the limits of sequences of rationals, Cauchy's remark was logically incomplete since he did not give a definition of irrational numbers. Bernard Bolzano (c. 1817) and C. F. Gauss were also thinking along this line.[2]

The persons responsible for the two chief theories of irrationals in use today are the Germans Georg Cantor (1845–1918) and J. W. R. Dedekind (1831–1916), both of whom published their first papers on this topic in 1872. Cantor's approach is somewhat similar to that propounded by the Frenchman C. Meray in 1869. His

[1] P. S. Jones, "Irrationals or Incommensurables I, II, III, IV," The Mathematics Teacher, XLIX (February, March, April, October, 1956), pp. 123–127, 187–191, 282–285, 469–471).

[2] Oskar Perron, *Irrationalzahlen* (1939), p. 55 ff. Although it has not been quoted directly, much use in this series has also been made of notes of J. Molk's translation exposition of A. Pringsheim's "Nombres Irrationnels et Notion de Limite" in Tome I, vol. 1 of *Encyclopédie des Sciences Mathématiques Pures et Appliquées* (Paris, 1904).

work is also similar, but less so, to the development of the German scholar, Karl Weierstrass, whose lectures of 1865–1866 on this topic were expanded and published by several of his students.

Cantor worked with sequences of rational numbers. Through definitions, he created a new number field out of the raw materials (rational numbers) which were on hand and understood. This process of extending old concepts or creating new ideas from old materials has happened repeatedly in the history of mathematics.

Cantor called a *fundamental sequence* any sequence of rational numbers $(a_1, a_2, a_3, \cdots, a_m, \cdots, a_n)$ which satisfied the condition that for every positive ϵ there was an N_ϵ such that $|a_m - a_n| < \epsilon$ for all $m \geqq N_\epsilon$, $n \geqq N_\epsilon$. Such sequences satisfy Cauchy's condition for convergence, and hence are sometimes today called *Cauchy*, *regular* or *convergent*. Perron calls this "the criterion of Bolzano-Cauchy-Cantor."

Cantor considered that every sequence of this type represented a *real* number. It is fairly easy to show that this new set of numbers, defined and represented by limits of regular sequences, contains within it at least one sequence for each rational number. For example the sequence, (1.9, 1.99, 1.999, $\cdots$, $\cdots$) formed in the obvious way corresponds to 2, while (.6, .66, .666, $\cdots$ $\cdots$) corresponds to 2/3.

Further, the sum of two regular sequences $(a_1, a_2, a_3, \cdots, a_n, \cdots) + (b_1, b_2, b_3, \cdots, b_n, \cdots)$ is defined to be $(a_1+b_1, a_2+b_2, a_3+b_3, \cdots, a_n+b_n, \cdots)$. This sum can be shown not only to possess the usual desirable properties of associativity and commutativity, but also demonstrates that the sum of the sequences representing two rationals will turn out to be a sequence representing that rational which is the sum of the two original rationals.

After similar definitions of the product of two sequences, it can be shown that if sequences of rationals are thought of as elements of a set, then this set, with the operations noted above, forms a number *field*. That is, addition and multiplication are closed, associative, and commutative; there is an identity element with respect to each operation; each element, except possibly zero, has an additive and multiplicative inverse; and multiplication is distributive with respect to addition.

Not only is the set of all regular sequences a field, but, as suggested above, there is a subset of these sequences which can be identified with the set of all rational numbers. Further, this subset also satisfies the field axioms in such a way that the result of adding or multiplying two rationals corresponds to the sequence which would result from adding or multiplying the sequences associated with the two original rationals. This isomorphism of the set of rationals with a subset of the set of regular sequences shows that the set of sequences is an *extension field* of the field of rationals.

Of course, just as there are many rationals corresponding to the integer 2 (e.g., 2/1, 4/2, 6/3, etc.), so there are many sequences corresponding to the rational 2/1. For example, in addition to that cited above, both (1, 3/2, 7/4, 15/8, $\cdots$) and (2, 2, 2, 2, $\cdots$) are such sequences. All of these are equal, however, under the definition of equality for regular sequences.

Before proceeding to a brief discussion of Dedekind's approach to irrationals, we should note several results related to Cantor's work. For example, the final solution of the ancient Greek problem of squaring the circle (with compasses and unmarked straight edge) was ultimately achieved only when Lindemann showed pi to be a transcendental number. Hence the problem is impossible of solution since ruler and compasses can construct only certain types of algebraic irrationals.

Euler had shown e to be irrational in 1737, long before Cantor; and Lambert had then shown pi to be irrational in 1767. However, some irrationals, such as the $\sqrt{2}$, are constructible, others such as

$\sqrt[3]{2}$, cannot be so constructed, and in particular none of that class of irrationals called "transcendental"[3] can be so constructed.

Although their existence had long been suspected, it was not until 1844 that Liouville proved that there are such things as transcendental numbers. Even then he did not produce a single particular example. In 1871 Cantor proved that there is an infinite number of transcendentals. To do this, his theory of transfinite numbers and infinite aggregates was needed. This theory showed as well that, although the transcendentals and rationals are both infinite in number, there are actually more transcendentals than there are rationals or even algebraic numbers.

The first number to be proven transcendental was e. Hermite demonstrated this in 1873. Lindemann's proof of the transcendence of pi followed in 1882.

In spite of the large number of transcendentals proven to exist, it has been and still is a difficult problem actually to identify them. In fact, the seventh of the famous twenty-three unsolved problems presented by David Hilbert in 1900 was to prove the transcendence of a class of numbers, of which $2^{\sqrt{2}}$ and e^{π} were examples. In 1929, the Russian, A. Gelfond, proved that e^{π} is transcendental, and in 1934 he gave a complete solution for Hilbert's problem, but the testing of particular numbers for transcendence is still not easy.[4]

Dedekind's extension of the rationals also defined new numbers by use of sets of the "old" rational numbers. However, instead of discussing convergent sequences, he talked of partitions of the set of all rationals into two sets, such that every number of the second set was greater than every member of the first. Every such partition was called a *schnitt*, or a *cut*, and by definition was identified with a real number.

In some cuts there is a last element in the first set, or a first element in the second. Cuts of this type are identified with the rational numbers. But there are also cuts that have no last element in the first set, nor first in the second. Such cuts may also be regarded as defining real numbers, but in these cases the numbers correspond to irrationals.

For example, if the first set includes all the rationals whose square is less than 2, it has no last element, and the second set containing all rationals whose square is greater than 2 has no first element. These two sets are a partition of the set of all rational numbers which defines the $\sqrt{2}$.

After defining addition, multiplication, and equality of cuts, they too can be shown to be a field which has a subset that is isomorphic to the rationals. Hence the set of cuts is an extension field of the rationals.

In teaching, there are two other related topics that ought to be mentioned with the study of incommensurables and irrationals. First, every rational number can be written as a periodic or repeating decimal fraction, while irrationals correspond to infinite nonrepeating decimals: Second, "quadratic surds" correspond to repeating continued fractions. The first fact should be the topic for an interesting discussion in many secondary school classes because the period of the decimal representation of rationals is associated with the use of the base 10, while the correspondence itself may be associated with geometric progressions in second-year algebra. Continued fractions are a less common topic which merit more attention as enrichment material associated with rationalizing radical expressions and the Euclidean algorism.

[3] Transcendental numbers are those which cannot be the root of any algebraic equation. Algebraic equations are all those which can be written in the form $a_nx^n+a_{n-1}x^{n-1}+\cdots ax+a=0$ where the exponents are integers and the a_i are rational numbers.

[4] Harry Pollard, *The Theory of Algebraic Numbers* (Mathematical Association of America, 1950) pp. 42–46; Einar Hille, "Gelfond's Solution of Hilbert's Seventh Problem," *American Mathematical Monthly*, XLIX (December, 1942), 654–661.

Edited by Adrian Struyk, Clifton High School, Clifton, New Jersey

Roots and logarithms

by Elizabeth F. Brown, Knoxville Junior High School, Pittsburgh, Pennsylvania

Any root—integral, fractional, or decimal—of any real number may be found to any desired degree of accuracy by using only the processes of elementary arithmetic and the laws of exponents, not the binomial theorem or Horner's method. The process is self-correcting in that accidental errors are automatically eliminated.

This method, as applied to square root, has been in use in a somewhat less generalized form since the time of Heron the Elder. Lindquist calls it the Carpenters' Rule. A similar method has been given in some recent textbooks, but in a more particular form that some pupils find to be a rule of many steps. It has also appeared in earlier issues of THE MATHEMATICS TEACHER.[1]

Since, when a number is divided by its square root the quotient is also the square root, then if the number be divided by something less than the root, the quotient must be more than the root. In other words, the square root of a number must lie between any divisor of that number and its quotient when the divisor and quotient differ. Therefore, to find the square root of a number: (1) divide the number by anything, (2) divide the number by something between the previous divisor and its quotient, (3) repeat Step 2 until the divisor equals the quotient to the desired number of decimal places or is 1 different in the last place. If the divisor is equal to the quotient, or is 1 more than the quotient in the last place, then the last divisor is the required root. If the divisor is 1 less than the quotient in the last place, the quotient is the root. But this part of the method is incidental and does not need to be remembered. Steps 1, 2, 3 are all that matter.

It may be seen that the closer the first divisor is to the required root the fewer divisions will be necessary.

Better guess = Less work

If each new divisor be the arithmetic mean between the preceding divisor and quotient, a further economy will be effected. It is not worth while to divide past the decimal point until the integral parts of the divisor and quotient are alike or only different by 1. Then the division may be carried to hundredths or thousandths, and on each succeeding division the number of digits in the quotient may be doubled. Bright pupils will soon find economical ways of applying this method, and slower pupils are not held back by a cumbersome rule.

[1] E. W. Chittenden, "On Square Root by Division," XXXIX (February 1946), 75–76; F. Gruenberger, "Square Root," XLIV (March 1951), 177–179; C. N. Shuster, "Approximate Square Roots," XLV (January 1952), 17–18; W. S. Tobey, "Square Root by Approximation and Division," XXXVIII (March 1945), 131–132; L. E. Ward, "On the Computation of Square Roots," XXIV (February 1931), 86–88.

Examples:

(a) To approximate $\sqrt{10}$

$$10 \div 3 = 3.33$$

$$10 \div 3.16 = 3.16455$$

$$10 \div 3.16227 = 3.16228$$

$$\sqrt{10} = 3.16228$$

(b) Find $\sqrt{15625}$

$$15625 \div 100 = 156$$

$$15625 \div 128 = 122$$

$$15625 \div 125 = 125$$

$$\sqrt{15625} = 125$$

(c) Find $\sqrt{8790}$ to nearest .001

$$8790 \div 90 = 97$$

$$8790 \div 93 = 94.516$$

$$8790 \div 93.758 = 93.751$$

$$8790 \div 93.754 = 93.755$$

$$\sqrt{8790} = 93.755$$

By extending the idea, higher roots may be found. If a number be divided by its nth root (n an integer), and that quotient be divided by the same divisor, and so on until there are $n-1$ divisions, then the final quotient will be the nth root. If the divisor be less (or more) than the nth root, then the $(n-1)$th quotient will be more (or less) than the nth root. So the nth root of a number must lie between any number used $n-1$ times as a divisor and the $(n-1)$th quotient, when the divisor and the last quotient differ by any amount.

To find the nth root of any number (n an integer): (1) divide the number by anything, (2) divide the quotient by the same divisor, (3) divide each succeeding quotient by the same divisor until there are $n-1$ divisions, (4) choose some number between the divisor and the last quotient as a new divisor, (5) perform another set of $n-1$ divisions using the new divisor, (6) repeat Steps 4 and 5 until the divisor is the same as the $(n-1)$th quotient or differs from it by less than $n-1$ in the last place. The last divisor is the required root.

For economy, use a careful estimate of the root for the first divisor. For the divisors in later sets of divisions, use the arithmetic mean of the $n-1$ equal divisors and the last quotient. Disregard all remainders until the last quotient in a set of divisions differs from the divisor by $n-1$ or less; then divide to the desired number of decimal places. If, however, a root is wanted to five or more places, get it correct to two or three places first, then extend the next set of divisions to the full number of places.

Examples:

(a) Find $\sqrt[3]{5832}$

$$15\,\underline{)5832}$$

$$15\,\underline{)\ 388}$$

$$25$$

Mean of 15, 15, 25 = 18

$$18\,\underline{)5832}$$

$$18\,\underline{)\ 324}$$

$$18$$

$$\sqrt[3]{5832} = 18$$

(b) Find $\sqrt[5]{2}$ to nearest .01

$$1.1\,\underline{)2}$$

$$1.1\,\underline{)1.81}$$

$$1.1\,\underline{)1.64}$$

$$1.1\,\underline{)1.49}$$

$$1.35$$

$$5\,\underline{)5.75}$$

$$1.15 = \text{new divisor}$$

$$1.15\,\underline{)2}$$

$$1.15\,\underline{)1.73}$$

$$1.15\,\underline{)1.51}$$

$$1.15\,\underline{)1.31}$$

$$1.13$$

$$\sqrt[5]{2} = 1.15$$

(e) Find $\sqrt[5]{0.156284}$ to nearest .001

$$
\begin{array}{l}
5\underline{/1562.84} \\
5\underline{/\ 312} \\
5\underline{/\ \ 62} \\
\quad 12 \\
\text{new divisor}=6 \\
6\underline{/1562.84} \\
6\underline{/\ 260} \\
6\underline{/\ \ 43} \\
\quad 7.16 \\
\text{new divisor}=6.29 \\
6.29\underline{/1562.84} \\
6.29\underline{/\ 248.16} \\
6.29\underline{/\ \ 39.50} \\
\quad 6.28 \\
\sqrt[5]{1562.84}=6.29 \\
\sqrt[5]{0.156284}=0.629
\end{array}
$$

If, as in the last example, the number is entirely decimal, with no figures to the left of the decimal point, the work is made easier, perhaps, by moving the point to the right n or $2n$ places—enough, that is, to have some digits to the left of the point. Then after finding the root of this new number, move the point in the root one or two places to the left.

A by-product of this study of roots is a method for solving the exponential equation $a^x=b$, where a and b are rational numbers. The equation $a^{2.315}=b$ means that

$$b=(a^2)(a^{.3})(a^{.01})(a^{.005}),$$

or

$$b=(a)^2(a^{.1})^3(a^{.01})^1(a^{.001})^5.$$

This is expressed in terms of roots by writing

$$b=a^2(\sqrt[10]{a})^3(\sqrt[100]{a})^1(\sqrt[1000]{a})^5.$$

Therefore, to solve $a^x=b$ for x: (1) Divide b by a, and the quotient by a, and succeeding quotients repeatedly by a, until the last quotient is less than a. The number of divisions is the integral part of x. (2) Divide the last quotient by $\sqrt[10]{a}$, and divide repeatedly until obtaining a quotient less than $\sqrt[10]{a}$. The number of divisions is the tenths digit of x. (3) Divide the last quotient of step 2 by $\sqrt[100]{a}$, and repeat until the quotient is less than $\sqrt[100]{a}$. The number of divisions is the hundredths digit of x. (4) Use the $\sqrt[1000]{a}$ in the same way, and so on.

Example:

$$10^x=756.3$$

(1) After two divisions by 10 the quotient is 7.563.

$$10^{2+}=756.3$$

(2) Divide now by $1.259=\sqrt[10]{10}$, with 7.563 as the first dividend. After eight divisions the quotient is 1.199.

$$10^{2.8+}=756.3$$

(3) Divide 1.199 by 1.023 ($=\sqrt[100]{10}$). After seven divisions the quotient is 1.021.

$$10^{2.87+}=756.3$$

(4) The last step involves nine divisions by $1.002=\sqrt[1000]{10}$.

$$10^{2.879}=756.3$$

It will be noticed that when $a=10$ in $a^x=b$, the value of x is the common logarithm of b. And so the process gives a method for finding these logarithms directly as powers of 10, without reference to any other numbers (such as e and the modulus) or formulas.

The defining equation for the common logarithm, log N, of a number N is

$$10^{\log N}=N.$$

Since this is merely a special form of the equation $a^x=b$ considered above, the calculation of common logs is the same as solving $a^x=b$ for x when a and b are given. Substitute 10 for a, N for b, and log N for x in the procedure just described.

Some observations on application of the process are as follows. If N is less than 10, but greater than 1, start with Step 2. The characteristic of the log is then 0. If N is less than 1, multiply it by a power of 10 to obtain a new N between 1 and 10. Proceed as before. When finished, subtract the number used as exponent of 10 from the logarithm as found. Here is a short cut: in Step 2, if the starting dividend is large, divide by $\sqrt{10}$, counting it as five divisions, or by $\sqrt[5]{10}$, counting two divisions; similarly in Step 3, $\sqrt[20]{10}$ and $\sqrt[50]{10}$ may be used, counting as five and two divisions respectively.

Note that this process of calculating logarithms is in logical succession to subtraction and division. Subtraction may be regarded as counting backwards; division is counting the number of subtractions. Finding a log is counting the number of divisions.

Here then is a way to introduce the idea of logarithms so that their reality as exponents is learned from experience, not merely presented as a theory. It is a method of calculating logarithms in such fashion that the student may come to see that logarithms are not just mysterious numbers in a "log book," but are really exponents in the equation $10^x = N$, and as such are subject to the ordinary laws of exponents.

Appended is a short table of roots of 10 useful for finding common logs to three places (values of r and T in $\sqrt[r]{10} = T$)

$r=$ 2	5	10
$T=$3.162	1.585	1.259
$r=$ 20	50	100
$T=$1.122	1.047	1.023
$r=$ 200	500	1000
$T=$1.011	1.005	1.002

What's new?

BOOKS

Secondary

Arithmetic at Work, Book 1, Howard F. Fehr, Veryl Schult, Boston, D. C. Heath and Company, 1955. Cloth, v+457 pp., $2.80.

Arithmetic in Life, Book 2, Howard F. Fehr, Veryl Schult, Boston, D. C. Heath and Company, 1956. Cloth, v+422 pp., $3.00.

College

Einfügrung in Die Hönhere Mathematics, Phil. Karl Strubecker, Munchen, Germany, R. Oldenbourg, 1956. Cloth, xv+821 pp., DM 36, —

Infinite Sequences and Series, (Dover edition), Konrad Knopp, translated by Frederick Bagemihl, New York, Dover Publications, Inc., 1956. Paper iii+186 pp., $1.75 (Cloth, $3.50).

Introductory Calculus with Analytic Geometry, Edward G. Begle, New York, Henry Holt and Company, 1954. Cloth, x+304 pp., $5.00.

Numerical Analysis, Volume VI, Proceedings of the Sixth Symposium in Applied Mathematics of the American Mathematical Society, edited by John H. Curtiss, New York, McGraw-Hill Book Company, Inc., 1956. Cloth, vi+303 pp., $9.75.

Miscellaneous

Atoms and Energy, H. S. W. Massey, New York, Philosophical Library, Inc., 1956. Cloth, 174 pp., $4.75.

A Treatise on Surveying (6th Edition), Volumes I and II, General Editor, W. Fisher Cassie, New York, Philosophical Library, Inc., 1956. Cloth, $20.00 per set.

Basic Mathematics Simplified (2nd Edition), C. Thomas Olivo, Albany, New York, Delmar Publishers, Inc., 1953. Cloth, ix+421 pp., $4.25.

Electrical Interference, A. P. Hale, New York, Philosophical Library, Inc., 1956. Cloth, vii+122 pp., $4.75.

Engineering Geometry and Graphics, Hollie W. Shupe, Paul E. Machovina, New York, McGraw-Hill Book Company, Inc., 1956. Cloth, vii+347 pp., $5.25.

Edited by Lucien B. Kinney and Dan T. Dawson, Stanford University, Stanford, California

Teaching the formula for circle area

by Jen Jenkins, Bethany College, Lindborg, Kansas

To make the concept of the formula for the area of the circle an old and accepted one in the minds of gangling, trying-not-to-be-schooled junior high youngsters of today is the goal of many an arithmetic teacher. How far short of his goal he falls is shown in the blank stares the high school teacher faces when he asks his algebra class or geometry class to state the formula.

It is an accepted fact that concept-building calls for a combination of real experiences and abstract thinking. How, then, can the junior high teacher make vivid and memorable the situation dealing with the learning of this area formula?

Many method books say, "Use workshop techniques in teaching mathematics in the junior high school," and then leave the novice to his own devices to determine how the workshop techniques should function.

Teaching of the formula for the area of a circle has too often been a matter of merely telling the pupils what to do. "Square the radius; take that answer times pi," we say. Over and over again the pupils follow this procedure, but this blind following of procedure guarantees no remembrance of it. The procedure need not be meaningless, and it need not be forgotten if the pupils are given experiences in which they see and do. There are figures to be seen—there are figures to be made and to be felt. Such seeing and doing bring about lasting visual and muscular impressions.

First, the pupils can make circles from graph paper, each pupil making two or more of different sizes, if the class is small. A radius of seven should be used for at least one of the circles; certainly it should be the first circle drawn. In larger classes only the teacher and several pupils in the front of the room may do the manipulating, but whatever is going on must be in sight of every pupil in the room.

By this time, the pupils should know that the number of square units in a surface is called the area of that surface. When asked by the teacher to find the number of graph paper units in the circle, the pupils naturally think of counting them. This may be accepted by the teacher as a method consistent with their level of understanding. A dot is used to indicate each square unit which is not less than a half square. Counting the dots will give an approximation to the number of square units which represent the area.

A shorter, quicker way, which will relegate the counting method to the status of "horse-and-buggy" procedure, is now explained. In naming the second method, the terms "atomic age" or "air age" may be employed. After the formula is worked out

and its use becomes easy, the title will seem to be very fitting when the two methods are contrasted. The rearrangement of the area of the circles into large squares is the basic idea of the formula.

The teacher asks each pupil to look at one of his circles. The questions and answers go something like this:

TEACHER: What two things do you know about your circle by glancing quickly at it?

PUPIL: I know where the center is. I know what the radius is.

TEACHER: Both the radius and the center are important to us in this exercise. We are going to cut up the circle and rearrange it into large squares. Since the radius and center are so very important, what would you guess the first large square we make would have for its center and for its side?

PUPIL: The first square will have the center of the circle for its center and the radius of the circle for its side.

TEACHER: Cut out from the center of your circle a square which has these characteristics. Watch me cut mine first. Be sure you count carefully. . . . Lay your square carefully on your desk. I will place mine on the flannelgraph so that you all may see. . . . Now, let's try to get another square the same size. . . . Yes, we will have to piece it. We will have to cut out four pieces for this square. Watch me cut my four pieces, one from each side. . . . See how they fit together to make another square. . . . Can we get a third square? . . . Let's try. It will take much piecing and we will not have square corners, but we will make it look the best possible. . . . There is our third square. . . .

PUPIL: I have some little pieces left.

TEACHER: They are too few to make a whole square, aren't they? How much of a square will they make? . . .

After the 3 1/7 squares (approximately) have been formed, the class should discuss the number. The statement, "The area of a square can be rearranged into about 3 1/7 squares which have the radius as a side," is placed on the blackboard. If the radius of 7 was used, the class will note that each large square has 7×7, or 49, small squares in it and that the three 49s plus the extra 1/7 of 49 equals 154. It would be well to compare this answer with the one obtained by counting.

A number of problems are worked out in which the pupils follow the word rule. Two or three days are required on the word rule and drawings such as the following:

The next step would be to cut down the word rule to formula size. The first cutting would result in something like this, "The area of a circle is 3 1/7 squares which have the radius for a side."

The class must seek for a short way to write the expression following "3 1/7." Questions like, "What did we take the 3 1/7 times?" lead the pupils to realize that it was always the radius times itself. Since they already know that 3 1/7 is π, they can complete the formula as soon as they learn the meaning of r^2. One way of doing this is to have them draw pictures of several problems, first using the letter r for the radius and placing r^2 in the squares instead of the number.

Often as review, the class may be asked to sketch a circle-area problem using both the number and the letter symbols, to set in their minds this most important formula. Thus, through visual and muscular impressions, the formula becomes to the pupil an old and accepted fact. It will remain in his mind where it will be truly understood.

The construction and measurement of angles with a steel tape: surveyor's method

by Richard W. Shoemaker, University of Toledo, Toledo, Ohio

Dear Dr. Kinney:

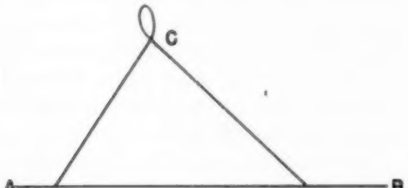
Reading Mr. Eagle's article in the TEACHER (December, 1955, pp. 563–564) brought to mind another method of constructing an angle with a steel tape. This surveyor's method, an account of which is enclosed, has several advantages and one disadvantage not enjoyed by Mr. Eagle's method.

In the surveyor's method it is not necessary to perform the awkward task of describing an arc of a circle of radius 57.3 feet, nor does it demand that the tape be bent around this arc in a manner contrary to the intended use of a tape. On the other hand, a table of sines and a smattering of trigonometry are required in the surveyor's method, and this may be a serious disadvantage at the junior high school level.

Sincerely,
Richard W. Shoemaker

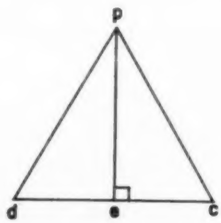
In the December 1955 issue of this journal, Mr. Eagle has shown how to construct an angle with a steel tape. There is another method, commonly known to most surveyors, of solving the problem.

Assume that at point P, on line segment AB, an $\angle APC$ equal to x degrees is to be determined. With the 100-foot mark of the tape on P, locate point D on



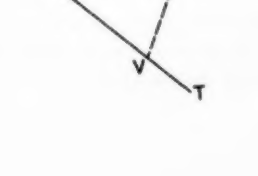
AP, such that $PD=50$ feet. Mark this point with a stake or pin. Now with the 0-foot mark of the tape at point D and the 100-foot mark at P, loop the tape as shown so that $PC=50$ feet and $DC=100 \cdot \sin \frac{1}{2}x$. That is, with the $100 \cdot \sin \frac{1}{2}x$-foot mark firmly held over the 50-foot mark, pull so that lengths DC and PC are both taut. Points P and C determine a line such that $\angle APC = x$ degrees. If $100 \cdot \sin \frac{1}{2}x$ is much greater than 45 feet, it will be necessary to use two tapes rather than just one.

To justify this method, consider triangle dpc, where $dp=pc=50$ feet and $dc=100 \cdot \sin \frac{1}{2}x$. Drop a perpendicular



from p to dc, meeting dc in e. Now $de=ec=50 \cdot \sin \frac{1}{2}x$, and $\angle dpe = \angle epc$. But the sine of $\angle dpe$ equals $(50 \cdot \sin \frac{1}{2}x)/50$, or $\sin \frac{1}{2}x$. Hence $\angle dpe = \frac{1}{2}x$ or $\angle dpc$ equals x degrees.

This procedure, with slight modification, can be used to determine the size of an existing angle. Given $\angle RST$, place the 50-foot mark of the tape at point S and determine points U on RS, and V on ST such that $SU=SV=50$ feet. Mark points U and V with stakes or pins. Next measure distance UV. But UV equals $100 \cdot \sin \frac{1}{2}(\angle RST)$, or $\angle RST$ equals $2 \cdot \text{arc} \sin UV/100$.



Edited by William L. Schaaf, Department of Education, Brooklyn College, Brooklyn, New York

Mathematical Handwriting on the wall. . .

It is not always given to human beings to live, as we do today, in an era of transition, trembling on the threshhold of an unknown Tomorrow. Nor is it an altogether comfortable feeling. From time to time a Huxley foretells *A Brave New World* or a George Orwell gives us his *Nineteen Eighty-Four*. Certain recent and not unrelated developments in mathematics may well furnish a clue to the future. I refer to the contributions of Nicolas Rashevsky and others to the creation of a *mathematical sociology*; to the work of von Neumann and Morgenstern; and to the science of *cybernetics*, heralded by Norbert Wiener, and more widely known as communication processes in animals and machines.

To a degree at least, the man on the street is already familiar with "automation," "feed-back," "servo-mechanisms," "machines-that-think," and the like. It is my considered belief that the socio-economic and political implications of the portents of these mathematical achievements may prove of the utmost significance if we will but read the handwriting on the wall.

The leading monographs[1] in these areas are burdened with a goodly amount of forbidding mathematics, even for many of the readers of this journal.

It would seem appropriate, therefore, for this Department to pass on to its readers some interpretation, however humble and inadequate, of the findings of these pioneer thinkers. We begin by calling attention to the phenomenon of *homeostasis*, a highly complex set of internal physiological control-mechanisms which automatically tend to maintain the life of a healthy, living organism. They include, for example, bodily temperature, blood pressure, hydrogen-ion concentration of the blood, rate of calcium metabolism, and many others. This concept of homeostatic processes may be applied to a society, considered as an organic entity. In the opinion of Norbert Wiener:[2]

> . . . one of the most surprising facts about the body politic is its extreme lack of efficient homeostatic processes. There is a belief, current in many countries, which has been elevated to the rank of an official article of faith in the United States, that free competition is itself a homeostatic process: that in a free market, the individual selfishness of the bargainers, each seeking to sell as high and buy as low as possible, will result in the end in a stable dynamics of prices, and will redound to the greatest common good. This is associated with the very comforting view that the individual entrepreneur, in seeking to forward his own interest, is in some manner a public benefactor, and has thus earned the great rewards with which society has showered him. Unfortunately, the evidence, such as it is, is against this simple-minded theory.

Wiener goes on to point out that our so-called free market is in effect a *game*, and as such, is amenable to the mathematical theory of games, strategy, and economic behavior as developed by von Neumann, Morgenstern, and others.

> This theory is based on the assumption that each player, at every stage, in view of the

[1] Nicolas Rashevsky, *Mathematical Theory of Human Relations* (Bloomington, Indiana: Principia Press, 1947); Nicolas Rashevsky, *Mathematical Biology of Social Behavior* (Chicago: University of Chicago Press, 1951); John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior* (Princeton: Princeton University Press, 1948); Norbert Wiener, *Cybernetics* (New York: John Wiley and Sons 1948).

[2] *Cybernetics*, p. 185 ff.

information then available to him, plays in accordance with a completely intelligent policy, which will in the end assure him of the greatest possible expectation of reward. It is thus the market game as played between perfectly intelligent, perfectly ruthless operators. Even in the case of two players, the theory is complicated, although it often leads to the choice of a definite line of play. In many cases, however, where there are three players, and in the overwhelming majority of cases, when the number of players is large, the result is one of extreme indeterminacy and instability. The individual players are compelled by their own cupidity to form coalitions; but these coalitions do not generally establish themselves in any single, determinate way, and usually terminate in a welter of betrayal, turncoatism, and deception, which is only too true a picture of the higher business life, or the closely related lives of politics, diplomacy, and war. In the long run, even the most brilliant and unprincipled huckster must expect ruin; but let the hucksters become tired of this, and agree to live in peace with one another, and the great rewards are reserved for the one who watches for an opportune time to break his agreement and betray his companions. There is no homeostatis whatever. We are involved in the business cycles of boom and failure, in the successions of dictatorship and revolution, in the wars which everyone loses, which are so real a feature of modern times.

Naturally, von Neumann's picture of the player as a completely intelligent, completely ruthless person, is an abstraction and a perversion of the facts. It is rare to find a large number of thoroughly clever and unprincipled persons playing a game together. Where the knaves assemble, there will always be fools; and where the fools are present in sufficient numbers, they offer a more profitable object of exploitation for the knaves. The psychology of the fool has become a subject well worth the serious attention of the knaves. Instead of looking out for his own ultimate interest, after the fashion of von Neumann's gamesters, the fool operates in a manner which, by and large, is as predictable as the struggles of a rat in a maze. *This* policy of lies—or rather, of statements irrelevent to the truth—will make him buy a particular brand of cigarettes; *that* policy will, or so the party hopes, induce him to vote for a particular candidate—any candidate—or to joint in a political witch hunt. A certain precise mixture of religion, pornography, and pseudo-science will sell an illustrated newspaper. A certain blend of wheedling, bribery, and intimidation will induce a young scientist to work on guided missiles or the atomic bomb. To determine these, we have our machinery of radio fan-ratings, straw votes, opinion samplings, and other psychological investigations with the common man as their object; and there are always the statisticians, sociologists, and economists available to sell their services to these undertakings.

Of all the anti-homeostatic factors in society, continues Wiener, the most effective and most significant is the control of the means of communication. This refers, of course, to books, newspapers, the telephone and telegraph, radio, television, the theater, the movies, schools, and the church. In a competitive society, based on buying and selling, these various media of mass communication contain within themselves the seeds of their own self-defeat. Instead of enhancing social homeostasis, communication facilities are soon constricted by those who seek self-aggrandizement rather than the well-being of society—the dictators, the hucksters, the demagogues.

Elsewhere Wiener points out that there are those who believe that man's control over his physical environment has outrun his control over his social environment:

Therefore, they consider that the main task of the immediate future is to extend to the fields of anthropology, of sociology, of economics, the methods of the natural sciences, in the hope of achieving a like measure of success in the social fields. From believing this necessary, they come to believe it possible. In this, I maintain, they show an excessive optimism, and a misunderstanding of the nature of all scientific achievement.

The social consequences of cybernetics, of automation, of mathematical game-theory, and of related developments are set forth in very readable yet incisive fashion in Wiener's *The Human Use of Human Beings*[3]. The next two passages will convey, at least in part, the burden of this all-important message:[4]

Those of us who have contributed to the new science of cybernetics thus stand in a moral position which is, to say the least, not very comfortable. We have contributed to the initiation of a new science which, as I have said, embraces technical developments with great possibilities for good and for evil. We can only hand it over into the world that exists about us, and this is the world of Belsen and Hiroshima. We do not even have the choice of suppressing these new technical developments. They belong to the age, and the most any of us can do by

[3] Houghton Mifflin, 1950.
[4] *Cybernetics*, pp. 38–39.

suppression is to put the development of the subject into the hands of the most irresponsible and most venal of our engineers. The best we can do is to see that a large public understands the trend and the bearing of the present work, and to confine our personal efforts to those fields, such as physiology and psychology, most remote from war and exploitation. As we have seen, there are those who hope that the good of a better understanding of man and society which is offered by this new field of work may anticipate and outweigh the incidental contribution we are making to the concentration of power (which is always concentrated, by its very conditions of existence, in the hands of the most unscrupulous). I write in 1947, and I am compelled to say that it is a very slight hope.

This final excerpt is taken from a discussion of the impact on society of the future development of automation systems and communication machines:[5]

I have said that the modern man, and especially the modern American, however much "know-how" he may have, has very little "know-what." He will accept the superior dexterity of the machine-made decisions without too much inquiry as to the motives and principles behind these. In doing so, he will put himself sooner or later in the position of the father in W. W. Jacobs' *The Monkey's Paw*, who has wished for a hundred pounds, only to find at his door the agent of the company for which his son works, tendering him one hundred pounds as a consolation for his son's death at the factory. Or again, he may do it in the way of the Arab fisherman in the *One Thousand and One Nights*, when he broke the Seal of Solomon on the lid of the bottle which contained the angry djinnee.

Let us remember that there are game-playing machines both of the Monkey's Paw type and of the type of the Bottled Djinnee. Any machine constructed for the purpose of making decisions, if it does not possess the power of learning, will be completely literal-minded. Woe to us if we let it decide our conduct, unless we have previously examined the laws of its action, and know fully that its conduct will be carried out on principles acceptable to us! On the other hand, the machine like the djinnee, which can learn and can make decisions on the basis of its learning, will in no way be obliged to make such decisions as we should have made, or as will be acceptable to us. For the man who is not aware of this, to throw the problem of his responsibility on the machine, whether it can learn or not, is to cast his responsibility to the winds, and to find it coming back seated on the whirlwind.

I have spoken of machines, but not only of machines having brains of brass and thews of iron. When human atoms are knit into an organization in which they are used, not in their full right as responsible human beings, but as cogs and levers and rods, it matters little that their raw material is flesh and blood. *What is used as an element in a machine, is an element in the machine.* Whether we entrust our decisions to machines of metal, or to those machines of flesh and blood which are bureaus and vast laboratories and armies and corporations, we shall never receive the right answers to our questions unless we ask the right questions. The *Monkey's Paw* of skin and bone is quite as deadly as anything cast out of steel and iron. The djinnee which is a unifying figure of speech for a whole corporation is just as fearsome as if it were a glorified conjuring trick.

The hour is very late, and the choice of good and evil knocks at our door.

[5] *The Human Use of Human Beings*, pp. 212–213.

Mathematical talent and the National Science Fair...

We hear a great deal these days about challenging the bright pupil in mathematics. Evidently the brightest pupils don't need to be challenged—they would seem to have little trouble in finding something intriguing, to judge by the showing made in the Seventh National Science Fair conducted by Science Clubs of America in Oklahoma City in May, 1956. An examination of the catalog of exhibits reveals that of 213 entries listed, no fewer than fourteen dealt with mathematics in one way or another. To be sure, there is something of an overemphasis on electronic computation, but what else is to be expected at this moment of history-in-the-making?

Believing that mathematics teachers may well be interested in the nature of the themes selected by these precocious teenagers, we list below the exhibits in question, giving the titles exactly as entered (our numbering):

1. Function Before Fashion: Mathematical Designs
2. The Five Regular Polyhedra in Nature
3. An Electronic Binary Computer
4. Electric Binary Counter
5. Binary Decimal Translator
6. Trigonometric Functions in the Unit Circle.
7. A New Three-Dimensional Analysis of the Geometric Series

8. An Electronic Computer
9. Symbolic Logic Computer
10. Remote Controlled Cybernetics Machine
11. High Speed Electronic Counting and Adding Machine
12. The Turing Computer
13. Analog Computer
14. Tetrahedron Tower

We note also that these fourteen youthful exhibitors came from high schools in twelve different states.

On preparing mathematical material for print. . .

On two previous occasions[6] we have alluded in these columns to one aspect or another of the art of printing mathematical material. Within the past year or so, two other items along these lines have appeared, only in these instances the emphasis is on the preparation of manuscript by authors of articles, monographs, or textbooks on mathematics.

The first is the excellent, comprehensive article by Cecil B. Read,[7] which by this time many of our readers will doubtless have seen, entitled "A Style Guide for Typewritten Mathematical Manuscripts." This is a very practical discussion, meticulous in detail, which should prove of great help even to more-or-less experienced authors. There are, in addition, many useful suggestions for *typing* mathematical material as well as for the preparation of the original copy. It would seem to us that this paper is well worth issuing as a reprint, or "separate."

The other item is an attractive monograph called *The McGraw-Hill Author's Book,*[8] which may not as yet have come to the attention of all who might be interested. It is not altogether new, since it appeared first in 1926, and has been revised half a dozen times since. It is quite general in scope, being addressed to authors in all fields of endeavor. However, the section on mathematical material (pages 20–28) is worth while.

In leaving this matter of printing mathematical material, it may not be inappropriate to call attention to two government publications which are related. The *Style Manual* (Abridged) issued by the U. S. Government Printing Office (revised, January 1945; 50¢), pages 103–108, fully describes the practice with regard to writing and printing numerals. An eight-page leaflet entitled *Military Standard Mathematical Symbols* (U. S. Government Printing Office, Washington, D. C., 1951; 15¢), deals with the standardization of symbols in arithmetic, algebra, elementary geometry, analytic geometry, trigonometry, elementary calculus, and analysis.

[6] See THE MATHEMATICS TEACHER, XLVIII (March 1955), pp. 165–166; also, XLIX (February 1956), 140–141.

[7] *School Science and Mathematics*, LV (October 1955), pp. 550–562.

[8] New York: McGraw-Hill Book Company, 1955; 88 p.

HAVE YOU READ?

CROSS, ROSAMOND, "The Basic Requirements of a Superior Education," *Journal of the American Association of University Women*, May 1956, pp. 218–221.

Although this article is not directed towards mathematics, it does give a framework within which we, as mathematics teachers, can fit our subject. You may disagree with the definition of "identical opportunity for all" or with the philosophy expressed in the article concerning the worth of the high school diploma and the A.B. degree. You will agree, I think, with the ideas on the basic requirements of a superior education, which includes the mastery of the basic tools of learning; understanding, the steady growth of competence, the accurate use of vocabulary, and the learning vehicles devised by teachers who challenge the thinking of the student. You will also be interested to note the indicated part which parents should play in providing a superior education.—PHILIP PEAK, *Indiana University, Bloomington, Indiana*

A column of unofficial comment

What's new in mathematics?

by F. Lynwood Wren, George Peabody College for Teachers, Nashville, Tennessee

The year 1830 is a date of great significance not only in the domain of mathematical thought but throughout the entire realm of critical inquiry, scientific investigation, and logical deduction. It was during this year that Lobachevsky made his startling announcement that there *does exist* another geometry just as valid as Euclid's. He had discovered a consistent system of geometrical thought logically derived from a system of postulates that differed from those of Euclid only in a fundamental contradiction of the postulate of parallelism. This announcement was strengthened by the virtually contemporaneous (1832) announcement by Bolyai of the independent discovery of an equivalent geometry based on the same contradiction of Euclid's parallel postulate. Further support came in 1854, when Riemann announced still another consistent system of geometry based on an entirely different contradiction of the parallel postulate.

These tradition-shattering pronouncements constituted a cataclysmic event in the history of thought, not so much because they opened up vast expanses for significant development in geometry, but because they freed the mind of man to reject the evidence of his senses for the sake of what his mind might produce. Such freedom has given us today a pattern of thinking in which the conventional is no longer inviolable, the absurd may well become respectable, the absolute is subject to dispute, and the obvious has become suspect.

Oriented in such a framework, mathematical thought during recent years has experienced a somewhat explosive emancipation. No longer does a system of axioms obey the dictates of the conventional and constitute a body of "self-evident truths." Rather, systems of mathematical axioms now are regarded merely "as nothing more than the imaginative creations of the human mind, working in the medium of concept and symbol" and under no other absolute control than that of consistency.

Operating within such a structure of freedom, mathematics, during the first half of the twentieth century has experienced a marvelous and somewhat incredible metamorphism. The foundations have been, and continue to be, subjected to critical examination. Some traditional techniques have been discarded as no longer efficient, others have undergone radical modification. Familiar definitions of many basic concepts have been dropped as passé and inexact, to be replaced by qualifications more modern in import. New structures have been postulated and subjected to critical examination, some to be discarded, some to be modified, some to be retained. Many new concepts and modern techniques are now considered as significant common property among mathematicians. Algebra has become just as distinctly and as clearly a domain of postulational thought as is geometry, and is considered by many to be more clearly so, and much more easily comprehended by less mature thinkers than is the subject matter of geometry. Non-Euclidean

geometries have been accepted as being just as respectable and significant as the conventional Euclidean geometry, while analysis has broken the bonds of traditionalism to put many of its techniques on a strictly axiomatic foundation.

This emergence into the abstract of the fields of geometry, algebra, and analysis, and the concomitant axiomatic synthesis of different systems, has given new significance to the question "What's new in mathematics?" No longer is this a question whose principal import is the reflection of the layman's conventional puzzlement at the possibility of finding anything new within a subject area so generally accepted as characteristically static in its structure. Rather, it now somewhat reflects a feeling of insecurity and helpless wonderment in informed persons who seek to keep fairly well abreast of developments within a subject area so generally recognized as characteristically dynamic in structure.

Finding the answer to this question is difficult even for the specialist in mathematics. He can, however, by discriminating selection from the vast field of mathematical literature, study those materials which keep him well informed in the domain of his own specialty and basically enlightened in the comprehensible fringes of significantly related areas.

At present, the difficulty of providing the teacher of secondary mathematics with a satisfactory answer to this all-important question is one of maximum proportions. The very nature of his professional responsibilities prevents this teacher from being a highly trained specialist within any restricted area of mathematical thought. His training must be broad. On the other hand, he should have the opportunity to become informed concerning the fundamental, but not so highly technical, properties and techniques of modern mathematics. His training should impress him with the fact that there is a "new look" in mathematics molded in the fashion of twentieth-century thinking. It should not leave him with a mere superficial acquaintance with such developments, nor should it overlook or destroy the vital tie with the heritage of past ages. It should furnish him with a firm conviction that he is informed in the fundamentals and should provide him with a mechanism for keeping in touch with those new developments that strengthen and invigorate the field of mathematics.

There is also a desperate need for new instructional materials in our texts and syllabi at the secondary level. What aspects of twentieth-century mathematics can be made intelligible at the level of mathematical maturity of the secondary pupil? This question is just as important as, and perhaps more troublesome than, the two questions: "What modifications should be made in the teacher-training program?" and "How can modern developments be interpreted for the nonspecialist?" These questions pose very important problems for the subject-matter specialist, the teacher-training specialist, the curriculum planner, and the textbook writer.

Certain important steps are being taken toward the solution of these problems. Organizations are studying the problems of curriculum structure and teacher training. Some "modernized" textbooks are beginning to appear on the market, and simplified interpretations of new developments are making their appearance in the literature.

Among the more significant efforts to write expository treatments of modern developments in mathematics is the forthcoming yearbook of the National Council, "Insights into Modern Mathematics." Here, specialists, selected for ability to write in expository style as well as for research activity in their special areas, describe many of the more fundamental aspects of modern mathematics. It is believed that this book will be a significant contribution in the direction of helping the teacher of secondary mathematics find an intelligible answer to the question, "What's new in mathematics?"

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Calculus, Differential and Integral, with Problems and Solutions, G. M. Peterson and R. F. Graesser, Littlefield, Adams and Co., Ames, Iowa, 1956. Paper, x+321 pp., $1.75.

A remarkably varied collection of completely solved problems constitutes the major portion of this book. Each problem is briefly discussed, and virtually every step in its solution is exhibited. Topics covered include not only ordinary differentiation and integration, but also partial differentiation, multiple integrals, infinite series, expansion of functions, hyperbolic functions, and polar coordinates. Many worthwhile applied problems are stated and solved. The problems are presented in excellent sequence, are well scaled as to difficulty, and illustrate adequately the various concepts considered.

In general, theoretical discussions are either entirely lacking, or are so condensed that they are inadequate for a clear presentation of the basic concepts of calculus. In Section 2.1 the derivative is defined as the limit of a difference quotient, with no geometric or physical relationships indicated. In Section 2.4 velocity is defined to be $s'(t)$ and acceleration is said to be $s''(t)$ with no attempt at amplification. Then in Section 3.1 the derivative $f'(x)$ is stated to be the slope of a curve $y=f(x)$ at any point (x, y), without recourse to any geometric argument. The uninitiated student has difficulty in tying together these concepts, and is provided with no help by the authors. Integration is treated more carefully, and is generally well done. Other topics, however, are merely mentioned and then illustrated by problems; topics treated in this manner include the derivative of a function of a function, the differential of arc length, area of a curved surface, natural logarithms, and multiple integrals.

In the opinion of the reviewer, this book should be very valuable to the student who desires a thorough review of the techniques of solving calculus problems. Students taking formal calculus courses should also find it very useful as a reference book, since the voluminous number of correctly solved problems serves to illustrate the general theory. Helpful techniques for solving problems may be acquired by noting the methods employed by the authors in this book.—*David D. Strebe, University of South Carolina.*

Elements of Mathematics, J. Houston Banks, New York, Allyn and Bacon, Inc., 1956. Cloth, x+422 pp., $5.75.

"No prequisite beyond elementary arithmetic is demanded of the reader," states the author of this book in the preface. He does admit that some knowledge of high-school algebra will help in certain chapters. Then, he proceeds to develop the fundamental concepts and techniques of mathematics, from counting and ordering, through the algebra of sets.

Number, proof, measurement, and function are the four phases around which the book is developed. These lead quite naturally to many concepts which, in the past, have usually been left for advanced study. One comes across terms such as denumerably infinite sets, ordered fields, Abelian groups, Dedekind axiom of continuity, transfinite numbers, and Boolean algebra. The traditional subject matter is not glossed over but treated comprehensively. Good exercises are included with answers provided for about half of them.

This book is a delightful one to read in retrospect and should be exciting for many readers not schooled in mathematics. However, the reviewer cannot help but feel that it would be difficult to use as a text for students with the minimum requirements indicated above. True, the text is organized so that almost any desired combination of chapters can be covered. Pruning the course would partially defeat the purpose for which the book was written, which is to give a fresh and modern treatment to classical subject matter. The author is to be commended for a well-written book which dares to depart from the traditional approach. It should be recommended reading for all teachers of junior-high and high-school mathematics as well as enterprising grade-school arithmetic instructors.—*C. H. Lindahl, Iowa State College.*

Functional Mathematics (Book 4), William A. Gager, Luther J. Bowman, Carl N. Shuster, Franklin W. Kokomoor, New York, Charles Scribner's Sons, 1956. Cloth, xiv+578 pp., $3.40.

This book for 12th grade is the fourth in a series of high-school textbooks which integrates algebra, arithmetic, geometry, trigonometry, statistics, analytic geometry, and calculus. By means of this integrated material, instead of the traditional sequence of separate subjects, the authors feel that mathematics becomes more

functional, in the sense that it functions in daily life. An abundance of interesting applications to common life problems helps to provide for transfer of training and thus to increase this functional quality of the mathematics presented. The book is intended to prepare students for college, or for life if they do not go to college.

The main sections of the book are devoted to: (1) the number system developed through a desire for a closed system, (2) proofs in plane geometry, (3) approximate numbers, (4) series (arithmetic, geometric, and binomial) with application to compound interest, installment buying, depreciation, and annuities, (5) life insurance, (6) exponents, logarithms, and slide rule, (7) trigonometry, (8) the function concept, (9) statistics, and (10) the calculus.

The treatment of these topics seems a bit uneven to the extent that the parts on the number system, approximate numbers, series, and slide rule carry the student along slowly and carefully, developing the topics in considerable detail, while the parts on geometry, trigonometry, and calculus seem to try to cover a great deal of ground somewhat superficially. In the case of the calculus, the superficial treatment is admitted and it is made clear that this is an introduction to some of the easy applications of calculus, to whet the students' appetite for a more serious study later. As such this part is well done.

The chapter on insurance is interesting and, of course, important to life, but it has practically no mathematics in it. The two chapters on statistics go farther than most high-school treatments of the subject, going as far as standard deviation and coefficient of correlation.

One of the specific good points of the book is the development of the exponential function e^x and its connection with the growth law. At some points in the geometry proofs somewhat longer than necessary are given and the student is urged to discover shorter proofs. Still another good part is the careful development of annuities, which are a good application of geometric progressions.

There are not many mathematical errors although two rather serious ones were noted. The words "real numbers" are used (without previously defining them) in defining "irrational numbers" and then real numbers are defined as "all rational and irrational numbers." Later a degree of *angle* is defined as "1/360 of a circle."

The tendency to state formulas or laws with no justification or explanation, which is present in the earlier books of the series, is still present, but to a lesser extent. Some teachers may not like the repeated introduction of a contrapositive theorem, just because it is a contrapositive, and not because it is of importance as a theorem. The occasional dependence on the earlier books in the series might make it awkward at times to use this book without having used the others. Possibly some work on probability, inequalities, or determinants would be more valuable mathematical training then the chapter on insurance.

The style of writing should be attractive to the students since the authors talk informally with them, asking many leading questions. The photographs are particularly interesting. The book is attractively bound, the print is large, and the pages are easy to read.—***Henry Swain**, New Trier Township High School, Winnetka, Illinois.*

Fundamental Concepts of Geometry, Bruce E. Meserve, Addison-Wesley Publishing Company, Inc., Cambridge 42, Mass., 1956. Cloth, ix+321 pp., $7.50.

During this period when there is national interest in special content courses in mathematics for secondary-school teachers, we are fortunate to have this text published. All recognize the value of providing for teachers an opportunity to become better acquainted with modern mathematical developments, especially as they are related to the content of secondary-school mathematics. *Fundamental Concepts of Geometry* makes an excellent contribution in this respect. This text would serve very satisfactorily for courses on foundations or fundamental concepts of geometry. It also can serve a valuable purpose as a reference book for teachers or as the basis for an in-service study group planned for mathematics teachers, with or without the assistance of an instructor.

One of the strengths of the text is the treatment of projective geometry. This reviewer belongs to the old school of thought which holds the point of view that a course in projective geometry can be a very valuable part of the training of a mathematics teacher. The integration of the traditional material of projective geometry into a longer treatment concerned with Euclidean plane geometry, and the more modern developments in this field, is in itself a major contribution. Somewhat reluctantly, a devotee of projective geometry must admit that this is a more reasonable approach to a course in geometry for teachers. Professor Meserve has wisely treated both synthetic and analytic projective geometry. His treatment of coordinants in the real projective plans is to be commended.

The inclusion of chapters on the foundations of geometry, the evolution of geometry, non-Euclidean geometry, and topology greatly strengthens the content of his book. The references to be found in Chapter I to logical systems and logical notations will be of assistance to many teachers who will read about various proposals for modernizing the secondary-school curriculum. The fact that this text is a companion volume to the earlier publication of Meserve's *Fundamental Concepts of Algebra*, adds to its importance.

If there were to be any adverse criticism, it might be on the attempt of the author to cover too much material in a text planned for a one-semester course. At a number of points, new ideas are introduced rapidly, and perhaps not given the attention that would be desirable

for the prospective teacher or in-service teacher. On the other hand, it can be recognized that the author had real difficulty in making his selection, and that the text is more usable because of its broad coverage of fundamental concepts. Some might feel that more exercises would be desirable. Again, the author doubtless had to choose between the length of sets of exercises and the great variety of topics to be covered.

There would be considerably less justification for present criticism of the teaching of secondary-school mathematics, and of the training of secondary-school mathematics teachers, if all teachers of geometry were to give some serious study in either formal class, or on their own, to this treatment of geometry.—*John R. Mayor, American Association for the Advancement of Science, Washington, D.C.*

Fundamental Mathematics, Thomas L. Wade, and Howard E. Taylor, New York, McGraw-Hill Book Company, Inc., 1956. Cloth, xiv+380 pp., $4.75.

This book is intended to serve two purposes: (1) to provide an adequate foundation for the usual first year college mathematics, and (2) to present the basic topics of mathematics needed for the general education of a well-informed person. It is expressly designed for the freshman-level college student.

The fundamental operations with integers (and with fractions) are logically developed, on the basis of the commutative, associative, and distributive laws. There is a smooth shift from arithmetic to the generalized algebraic symbolism. The authors have emphasized the abstractness and the logic of mathematics. This strength in algebra should fulfill the first purpose.

Data obtained from research by the reviewer were used to evaluate the content in terms of the second purpose. It was found that *Fundamental Mathematics* includes essentially all the topics having high index values in the areas of arithmetic, algebra, business, statistics, and trigonometry. In the area of geometry, many topics having high index values for the purposes of general education (e.g., latitude and longitude) are not included. There is relatively little historical or biographical material. The book is of evident superior quality for the second purpose.

Suggested assignments are made for a three-semester-hour course emphasizing the mathematics needed for general education, and also for a three semester hour course in preparation for the sequential mathematics. The problem sets appear to be well suited to the varying abilities that one finds in classes for which the book is designed. There are ample problems and exercises, including a collection of more than one-hundred miscellaneous problems at the end of the book. The teacher's manual contains helpful specific suggestions to instructors, as well as answers to even-numbered exercises.—*Lauren G. Woodby, Central Michigan College*

A Modern Introduction to Mathematics, John E. Freund, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1956. Cloth, xvi+543 pp., $6.00.

This book is intended for a liberal arts program and makes very few demands upon the student's high-school mathematics. Several aspects of contemporary mathematics are reflected in the point of view of the author and in the content of the book. Accordingly, this book is purposely quite different from traditional college-freshman texts intended for prospective engineers, and from texts which rely heavily upon the student's background in high-school mathematics.

The first quarter of the book is devoted to a development of natural numbers, integers, rational numbers, real numbers, complex numbers, and their properties. Although the approach is somewhat formal, through postulates and theorems, the exposition is very readable. Some readers will regret the absence of extensive sets of exercises. All readers should appreciate the discussion of number notations and the frequent use of historical details.

The book also includes introductions to progressions, systems of linear equations, the theory of groups, a finite geometry, analytic geometry, graphical methods of solution, trigonometry, polynomial calculus with applications, transfinite numbers, logic including the algebra of classes, probability and statistics, geometries of higher dimension, non-Euclidean geometry, and topology. Three appendices include introductions to traditional topics of exponents and logarithms, mathematical induction and the binomial theorem, and trigonometric identities and formulas.

The broad scope of the book and the elementary level of the exposition make it necessary for the treatments of the above topics to be brief and, at times, intuitive. A strong point of the book is its emphasis upon logical concepts and sound mathematical reasoning.

Most junior- and senior-high-school mathematics teachers would find this book a pleasant introduction to a contemporary view of elementary mathematics. The book should provide an opportunity for such teachers to broaden their understanding of the mathematical concepts that they teach.

College teachers will find that, as mentioned above, this is not a traditional text. It is for a particular curriculum—the liberal arts. For students who take only one year of college mathematics it provides a logically sound introduction to mathematics. For students preparing to continue their mathematical training, your reviewer is among those who regret the absence in this book of more extensive sets of exercises. For example, the exercises for the section on linear equations in one unknown consist of eight routine equations in one exercise and five word problems; the exercises for the section on quadratic equations consist of eighteen routine problems and seven word problems. The related

skills are not systematically developed in later exercises. Thus the book is an introduction to mathematics, as stated in its title; it does not contain sufficient exercises to enable most students to develop a mastery of the topics treated and thereby to prepare themselves for more advanced mathematical courses.

In the opinion of this reviewer the book should serve general students effectively and should be a very readable introductory book for high-school teachers who would like to increase their insight into the nature of mathematics, especially contemporary mathematics. —*Bruce E. Meserve, Montclair State Teachers College, Montclair, New Jersey.*

Using Mathematics, 7, Kenneth B. Henderson and Robert E. Pingry, New York, McGraw-Hill Company, Inc., 1956. Cloth, xi+436 pp., $2.96.

This is an attractive book which seventh grade pupils will enjoy. The approach is through questions asked the student, except where a fraction is multiplied by a fraction: In this case the rule is given first. There are banner topic headings regularly given such as "What Can You Do in Division?" and "What Do You Know in Division?". The text is well tied together by references to preceding work, by showing how addition is related to multiplication and how ratio, fractions and division are all similar.

Throughout the textbook the question "Do you see?" is used and then the answer is given by telling what should have been gotten from the explanation or from working the problems. Each chapter has an excellent self test which gives some practice in all the work covered to date. There is some repetition of problems in these self tests from the text material.

There is an abundance of arithmetic; the drill problem and the story problem. There are also questions which lead the pupil to think about how and why he works the problem, as well as just asking him to work it to get an answer. There are problems which require that the student decide what is in excess, or what more is needed, for solving the problem. Color is used to an advantage.

The reviewer was especially pleased to see that the chapter on decimals opened with the sentence "Decimals are fractions." Emphasis is placed on the idea of numbers compared by all the different methods such as subtraction, division, ratio, and so forth. In general, all directions, explanations and definitions are very clear except one. Page 134 asks for a common fraction to be changed to its equal decimal and carry to hundredths. In this case the student will probably wonder what to do with fractions like 1/3.

The summaries are good, giving the words, understandings, and skills. There are different kinds of measure but no work with varying precision. The material covered is standard and, in the opinion of the reviewer, this book could easily be taught so as to meet the needs of the range of students in grade 7.—*Philip Peak, Indiana University, Bloomington, Indiana.*

BOOKLET

Problems in Mathematical Education, Educational Testing Service, Princeton, New Jersey. 50-page booklet written by Henry S. Dyer, Robert Kalin, and Frederic M. Lord; $1.00.

This booklet, written by three members of the staff of the Educational Testing Service, is an outgrowth of a survey done by ETS at the request of the Carnegie Corporation of New York. ETS was asked to "describe the kinds of research activities which would lead to better mathematics courses and teaching in the elementary and secondary schools." (Preface, p. i.) The purpose of the booklet, according to the authors, is to "define the problems in terms of what has been written and said on the subject and to locate the areas that need the attention of research." (Preface, p. i.)

The booklet is organized into five sections: Problems in Mathematical Education, The Learner, The Teacher, The Curriculum, and The Big Need. A comprehensive bibliography of 250 items is included at the end of the booklet.

In the opinion of the reviewer, this booklet represents the result of a careful analysis of the literature in mathematical education and presents an outstanding formulation of the major problems to be overcome in this area of education. This booklet is so excellent that every person interested in improving the program of education in mathematics should read and study it.—*Richard D. Crumley.*

DEVICE

Space Spider, Walker Products, 1530 Campus Drive, Berkeley 8, California. Kit of three masonite sheets, each 5¾" square, three 10 yd. spools of colored, fluorescent, elastic thread; blunt needle, 10 black clips, and instruction booklet; $2.95.

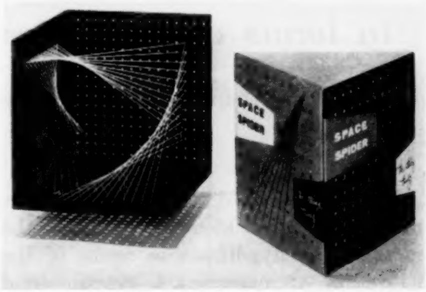
The *Space Spider*, when assembled, represents for the reviewer one octant of three-dimensional space, and thus permits the construction of some ruled surfaces as well as other three-dimensional figures. This is accomplished by threading the elastic cord through the holes found in each masonite sheet. Each panel has 225 holes spaced into 15 rows with 15 holes in each row. The holes are almost three eighths of an inch apart. Each panel has notches along two of its edges making the assembly of the device very easy. The panels are black and on the reverse side each has an index system of small letters (A through O) printed along one edge and small numerals (1 through 15) printed along a perpendicular edge. The instruction

booklet gives detailed directions for constructing nine different designs.

This device was designed for use as a toy, or for rehabilitation therapy. Since it may be used to represent a three-dimensional coordinate system, it can be quite useful in the teaching of mathematics. At the secondary school level, this device could be used as a means for stimulating interest in, and appreciation for, the beauty in geometric forms. The fact that some curved surfaces can be generated by a straight line can easily be discovered by a student. Many students will find this discovery fascinating. At the college level this device can be used with small groups to show the rectangular coordinate system for three dimensional space. A point in space can be represented by the head of a wooden safety match (the base of the match will fit snugly in any of the holes of the panels). Any line in space may be represented by elastic thread if the line intercepts two coordinate planes within the range of the device. There are no holes provided for points on either of the three coordinate axes, but these holes can be drilled if desired. It would be advisable in this case to glue the joints where the panels meet.

The possible uses of this device in the teaching of mathematics are limited only by the extent of one's creative ability.—*Richard D. Crumley*

Space Spider



"While the method of symbols is still far too widely used in practice, no educationist defends it; all condemn it. It is not, then, necessary to dwell upon it longer than to point out in the light of the previous discussion why it should be condemned. It treats number as an independent entity—as something apart from the mental activity which produces it; the natural genesis and use of number are ignored, and, as a result, the method is mechanical and artificial. It subordinates sense to symbol."—*Taken from* The Psychology of Number *by James A. McLennan and John Dewey. New York: D. Appleton and Company, 1916. Chapter IV, page 61.*

Teacher (finishing a long problem in algebra): "The result is $X = 0$."

A sleepy voice from the rear: "All that work for nothing."

Edited by Francis G. Lankford, Jr., Longwood College, Farmville, Virginia

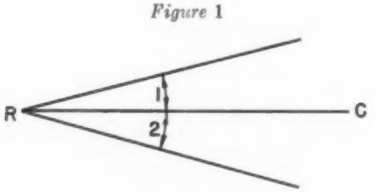
Interpretation of the hypothesis in terms of the figure

by Helen L. Garstens, Arlington County Public Schools, Arlington, Virginia

It is frequently difficult for the beginning geometry student to interpret the facts of the hypothesis in terms of the figure for an exercise. A practice used extensively to assist the student with this difficulty, consists of marking, in like-colored chalk, those parts of the diagram either explicitly stated as equal or inferred so by definition. Thus if *RC* bisects angle *R* in Figure 1, the student may mark angle 1 and angle 2 with the same number of like-colored arcs, because "bisects" means to divide into two equal parts. The advantages of such a technique seem obvious enough. However, even when the student knows that "bisects" means to divide into two equal parts, he may err in choosing the correct equal parts in the figure. Thus in Figure 2, if *BC* is given as the bisector of *DE*, often the beginning student is uncertain as to which of the line segments, *BC* or *DE*, is bisected, or whether both are bisected! He may therefore, incorrectly mark *BA* equal to *AC* or mark all four segments equal, though he is aware of the correct definition of "bisect."

It is my purpose here to indicate some examples of the exercises I have used to alert the student to the need for careful interpretation of the hypothesis and to the pitfalls of hasty judgments when marking his diagram. The exercises that follow have been found most effective when presented to students individually, and at judicious intervals, rather than all during a single lesson period. The correct answers are given here to help the busy beginning teacher in using these exercises.

Figure 1



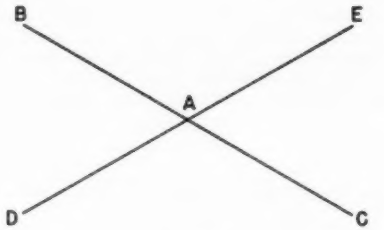


Figure 2

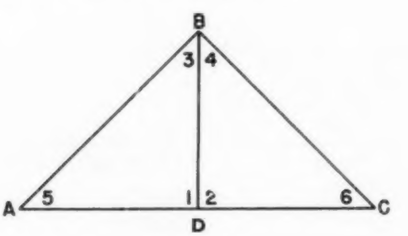


Figure 3

1. a. If *BD* bisects angle *B*, what relationship exists between angle 1 and angle 2? (See Fig. 3)

Answer: Cannot tell. Figure could be drawn like this (Fig. 4).

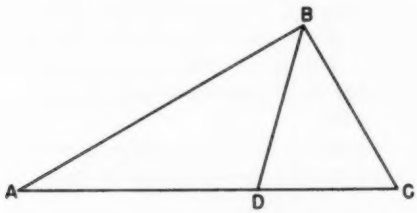


Figure 4

b. If BD bisects angle B, how does AD compare in size with DC?
Answer: Cannot tell.

c. If BD bisects angle B, what conclusion can you reach about AB and BC?
Answer: Cannot tell.

d. If BD bisects angle B, may we then say that angle 5 is equal to angle 6?
Answer: Cannot tell.

e. If BD bisects angle B, what can you conclude about angle 3 and angle 4?
Answer: Angle 3 equals angle 4.

2. a. If BD is perpendicular to AC, what relationship exists between angle 1 and angle 2?
Answer: Angle 1 equals angle 2 because both are right angles.

b. If BD is perpendicular to AC, how does AD compare in size with DC?
Answer: Cannot tell. Figure could be drawn like this (Fig. 5).

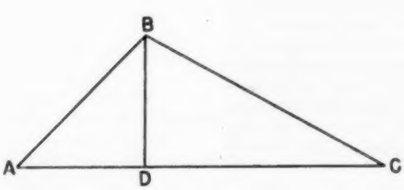


Figure 5

c. If BD is perpendicular to AC, does AB equal BC?
Answer: Cannot tell.

d. If BD is perpendicular to AC, what can you conclude about angle 3 and angle 4?
Answer: Cannot tell.

e. If BD is perpendicular to AC, how does this effect angle 5 and angle 6?
Answer: Cannot tell.

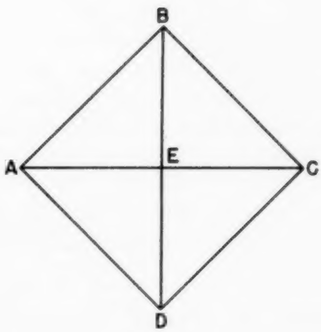


Figure 6

3. a. If C is equidistant from the endpoints of line segment BD, which segments may we mark equal in the diagram? (See Fig. 6).
Answer: BC and DC.

b. If C is equidistant from the endpoints of the line segment BD, how does BA compare in length with DA?
Answer: Cannot tell.

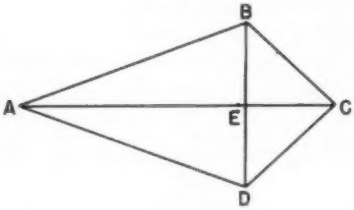
c. If C is equidistant from the endpoints of the line segment BD, does this tell us that DC equals DA? How might we modify this hypothesis so that DC might be marked equal to DA?
Answer: No; if D is equidistant from the endpoints of segment AC.

d. If AC is the perpendicular bisector of BD, how does BE compare in length with ED?
Answer: BE equals ED.

e. If AC is the perpendicular bisector of BD, how does AE compare in length with EC?

Answer: Cannot tell. Figure could be drawn like this (Fig. 7).

Figure 7



f. If AC is the perpendicular bisector of BD, is AC the bisector of BD or is BD the bisector of AC?
   Answer: AC is the bisector of BD.

g. If AC is the perpendicular bisector of BD, is AC the perpendicular to BD or is BD the perpendicular to AC?
   Answer: AC is perpendicular to BD and BD is perpendicular to AC.

"The mathematical analysis of spatial or temporal relation in nature, as Russell well knows, cannot be determined *a priori*. Mathematics furnishes us possibilities; it does not decide facts. Whether relations are discrete or continuous, finite or infinite, etc., must be settled by evidence. Mathematics is not concerned with the empirical world. After all, mathematics is entitled to its own world. There is nothing so admirable about the factual world. The mathematicians are not to blame if metaphysicians confuse types."—*Taken from* The Philosophy of Bertrand Russell. *Evanston, Illinois: The Library of Living Philosophers, Inc., 1946. P. 480.*

A program of mathematics for the junior high school which emphasizes problem-solving can be built around concrete problem situations. The mathematical content of such a program is very similar to the content recommended by authorities for a good mathematics program for these grades.—*This statement is based on the study by William Lee Carter.* A New Basis of Organization for the Junior High School Mathematics Program, *Ph.D., 1952, Ohio State University, Columbus; Major Faculty Adviser, Dr. Harold F. Fawcett.*

• **NOTES FROM THE WASHINGTON OFFICE**

Yearly financial report

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D.C.

Members of the Council will be glad to know that we have enjoyed another good year financially. The report at the end of the fiscal year, ending May 31, 1956, showed that our cash resources had increased in the amount of $8,818.28. A complete report of receipts and expenditures is given below.

You will note that the change in membership and subscription arrangements giving membership to *Arithmetic Teacher* subscribers and providing a reduced joint rate for those who wish both journals, has required a new breakdown for recording and reporting membership and subscription receipts. As would be expected, our

Receipts and Expenditures of The National Council of Teachers of Mathematics for the fiscal year, June 1, 1955—May 31, 1956

Receipts		
Memberships with THE MATHEMATICS TEACHER subscriptions	$27,580.26	
Memberships with *The Arithmetic Teacher* subscriptions	10,282.62	
Institutional subscriptions to THE MATHEMATICS TEACHER	13,371.44	
Institutional subscriptions to *The Arithmetic Teacher*	5,758.89	
Subscriptions to *The Mathematics Student Journal*	5,283.31	
Sale of advertising space in THE MATHEMATICS TEACHER	3,552.98	
Sale of advertising space in *The Arithmetic Teacher*	882.25	
Interest on U. S. Treasury Bonds	375.00	
Net profit from conventions	296.84	
Miscellanous	2.43	
Sale of publications		
Yearbooks	7,118.46	
Miscellaneous	11,407.84	
Total Receipts		$85,912.32
Expenditures		
Washington office	$27,498.51	
President's office	1,265.50	
THE MATHEMATICS TEACHER	24,188.77	
The Arithmetic Teacher	8,109.94	
The Mathematics Student Journal	2,832.26	
Committee work	715.78	
Travel by Board members	1,026.54	
Printing and preparation of supplementary publications	11,456.74	
Total expenditures		$77,094.04
Increase in Cash Resources		**$ 8,818.28**

largest source of income is from the sale of memberships and subscriptions, with the sale of publications ranking second and the sale of advertising space in the journals, third.

The greatest single expense is for the operation of the Washington Office. This expenditure covers all costs for salaries (for a staff of five or six persons), office forms and supplies, handling and mailing of publications sold, renewal notices, membership and sales promotion, and travel by the executive secretary. The second largest item of expense is for the editing, printing, and mailing of our official journal, THE MATHEMATICS TEACHER. Funds for the preparation and printing of supplementary publications rank third.

The cash profit we made during the 1955–56 fiscal year will give us a good cushion for helping to absorb two large bills during the present year. Our first directory of members of the Council since 1930, was printed during the summer and distributed to all current members; also, work is now under way on the printing of the 23rd Yearbook.

Members who have further questions are invited to write to the executive secretary at the Washington Office.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

CHRISTMAS MEETING

December 27–29, 1956
Arkansas State College, Jonesboro, Arkansas
Lyle J. Dixon, Arkansas State College, State College, Arkansas

ANNUAL MEETING

March 28–30, 1957
Bellevue-Stratford Hotel, Philadelphia, Pennsylvania
M. Albert Linton, William Penn Charter School, Philadelphia, Pennsylvania

JOINT MEETING WITH NEA AND NSTDA

July 1, 1957
Philadelphia, Pennsylvania
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

SUMMER MEETING

August 19–21, 1957
Carleton College, Northfield, Minnesota
Margaret Linster, St. Louis Park High School, Minneapolis 16, Minnesota, or Kenneth O. May, Carleton College, Northfield, Minnesota

Other professional dates

Chicago Elementary Teachers Mathematics Club
December 10, 1956
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Genevieve E. Johnson, Volta School, Chicago, Illinois

Women's Mathematics Club of Chicago and Vicinity
February 9, 1957
Tearoom of The Fair Store, State and Adams Streets, Chicago, Illinois
Ruth Woerner, 11715 S. 82nd Court, Palos Park, Illinois

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

NCTM Committees (1956-57)

An asterisk before an entry indicates a new appointee.

AFFILIATED GROUPS, 1957

Elizabeth Roudebush, Seattle, Washington, Chairman, 1959
Central: Adeline Riefling, St. Louis, Missouri, 1957
Southeastern: Houston Banks, Nashville, Tennessee, 1958
Northeastern: Catherine Lyons, Pittsburgh, Pennsylvania, 1958
Western: *Lesta Hoel, Portland, Oregon, 1959
Southwestern: *Eunice Lewis, Norman, Oklahoma, 1959
North Central: *Virginia Pratt, Omaha, Nebraska, 1959

AUDITING

Gwendolyn Holland, Washington, D. C., 1957
*Ethel H. Grubbs, Washington, D. C., 1958

Duties: To make an annual audit of the accounts of the National Office. Also to study and advise the Board on the value or necessity of having all accounts of the office audited by an authorized public accounting firm.

BUDGET

John R. Mayor, Madison, Wisconsin, Chairman, 1957
Houston Karnes, Baton Rouge, Louisiana, 1958
*Agnes Herbert, Baltimore, Maryland, 1959

Duties: To draw up and present to the Board the budget for the year 1957–58.

CLEVELAND CONVENTION, 1958

*Ona Kraft, Cleveland, Ohio, Chairman
Other members will be nominated by Cleveland and Ohio groups

CO-OPERATION WITH INDUSTRY, 1958

Bruce Meserve, Montclair, New Jersey, Chairman
Kenneth Brown, Washington, D. C.
William Glen, Pasadena, California
Phillip Jones, Ann Arbor, Michigan
*Zeke Loflin, Lafayette, Louisiana
W. W. Rankin, Durham, North Carolina
W. M. Whyburn, Chapel Hill, North Carolina
Marie Wilcox, Indianapolis, Indiana
*Lauren Woodby, Mt. Pleasant, Michigan

Duties: To continue work as reported to the Board at Milwaukee, April, 1956. Also to study ways and means that industry can aid mathematics teachers to attend conventions, summer institutes and to secure sabbatical leave.

COUNTING VOTES AND REPORTING ELECTIONS, 1957

F. Lynwood Wren, Nashville, Tennessee, Chairman
Myrl Ahrendt, Washington, D. C.
Maurice Hartung, Chicago, Illinois

Duties: To develop a procedure for sending out ballots, counting ballots, and reporting results of annual elections to the Board and Members.

CURRICULUM, ELEMENTARY

Fred J. Weaver, Boston, Massachusetts, Chairman, 1958
Joyce Benbrook, Houston, Texas, 1957
Esther Swenson, University, Alabama, 1957
Irene Sauble, Detroit, Michigan, 1958
*Ann Peters, Keene, New Hampshire, 1959
*Henry Van Engen, Cedar Falls, Iowa, 1959

Duties: To study the elementary school curriculum and report on this curriculum to the nation.

CURRICULUM, SECONDARY

Frank B. Allen, LaGrange, Illinois, Chairman
Howard F. Fehr, New York, New York
John Mayor, Madison, Wisconsin
Bruce Meserve, Montclair, New Jersey
Sheldon Myers, Princeton, New Jersey
*Alfred Putnam, Chicago, Illinois
*Marie Wilcox, Indianapolis, Indiana

Duties: To study the secondary school curriculum and report on this curriculum to the nation.

Evaluation Criteria for Middle States Association, 1957

Bruce Meserve, Montclair, New Jersey, Chairman
Marion Cliffe, Los Angeles, California
Donovan Johnson, Minneapolis, Minnesota
Philip Peak, Bloomington, Indiana
Max Sobel, Newark, New Jersey

Duties: To aid in revising evaluative criteria in mathematics for the Middle States Association. To co-operate with subcommittee of classroom teachers in testing the criteria, make report to the Board in 1957.

Executive, 1957

Howard F. Fehr, New York, New York, Chairman
Jackson Adkins, Exeter, New Hampshire
Phillip Jones, Ann Arbor, Michigan

Duties: To act for the Board on matters which need immediate action and to interpret policies of the Board.

International Relations

Henry W. Syer, Boston, Massachusetts, Chairman, 1957
Veryl Schult, Washington, D. C., 1957
*H. Behnke, Münster, Germany, 1958
*Alfred Putnam, Chicago, Illinois, 1958
*E. H. C. Hildebrandt, Evanston, Illinois, 1959
*Robert Rourke, Kent, Connecticut, 1959

Mathematics for the Talented, 1958

Daniel B. Lloyd, Washington, D. C., Chairman
Mary Lee Foster, Arkadelphia, Arkansas
Robert S. Fouch, Tallahassee, Florida
Francis Johnson, Oneonta, New York
*Glen Vanatta, Indianapolis, Indiana
*Robert Vollmer, Battle Creek, Michigan

Duties: To study and propose methods of selecting, preparing, and using materials for abler students. To propose policy for the NCTM in co-operating with agencies and groups concerned with better mathematical training for talented students.

Membership, 1957

Mary Rogers, Westfield, New Jersey, Chairman
Myrl Ahrendt, Washington, D. C.
Pearl Bond, Beaumont, Texas
Lucy Hall, Cheney, Kansas
Janet Height, Wakefield, Massachusetts
Harold Hunt, Seattle, Washington
Nelle Kitchens, Columbia, Mississippi
L. Clark Lay, Pasadena, California
Faith Novinger, Washington, D. C.
Bess Patton, Atlanta, Georgia
Mary Reed, Benton Harbor, Michigan

Duties: To strive to increase the membership to 15,000; keep itself and the Board informed on the membership growth by working through State Representatives; suggest ways and means of increasing membership; and to put on local drives where necessary.

National Council Representatives

AAAS. Co-operative Committee on Science and Mathematics
Henry W. Syer, Boston, Massachusetts, 1958

Education Advisory Committee to Science Service
Veryl Schult, Washington, D. C., 1958

Policy Committee of Mathematics Associations
John Mayor, Madison, Wisconsin, 1957

Subcommittee of International Committee on Mathematics Instruction
E. H. C. Hildebrandt, Evanston, Illinois, 1958
Henry Syer, Boston, Massachusetts, 1958

Nominations and Elections, 1957

F. Lynwood Wren, Nashville, Tennessee, Chairman
Alice Hach, Racine, Wisconsin
Maurice Hartung, Chicago, Illinois
Janet Height, Wakefield, Massachusetts
L. Clark Lay, Pasadena, California
Zeke Loflin, Lafayette, Louisiana
John Mayor, Madison, Wisconsin
Bess Patton, Atlanta, Georgia
Philip Peak, Bloomington, Indiana

Duties: To secure and recommend nominations for Spring election (1957). The committee is to nominate two persons for each of the offices: (1) Vice President, elementary school level, (2) Vice President, college level, and (3) three members for the Board of Directors.

Nominations and Elections, 1958

Clifford Bell, Los Angeles, California, Chairman
Jackson Adkins, Exeter, New Hampshire
Glenn Ayre, Macomb, Illinois
Mary Lee Foster, Arkadelphia, Arkansas
William Gager, Gainesville, Florida
Irene Sauble, Detroit, Michigan
Veryl Schult, Washington, D. C.
Marie Wilcox, Indianapolis, Indiana
F. Lynwood Wren, Nashville, Tennessee

Duties: To secure and recommend nominations for Spring election (1958). The committee nominates two persons for each of the offices: (1) President, (2) Vice President, elementary school level, (3) Vice President, secondary school level, and (4) three members of the Board of Directors.

Nomination of Editor of *The Arithmetic Teacher*, 1957–60

John R. Clark, New Hope, Pennsylvania, Chairman
Marguerite Brydegaard, San Diego, California
Charles Butler, Kalamazoo, Michigan

Foster Grossnickle, Jersey City, New Jersey

Duties: To nominate a number of people for editor of *The Arithmetic Teacher.*

OFFICE OF EXECUTIVE SECRETARY, 1957

John Meyor, Madison, Wisconsin, Chairman
Jackson Adkins, Exeter, New Hampshire
Glenn Ayre, Macomb, Illinois
Veryl Schult, Washington, D. C.

Duties: To recommend persons for the office for the term 1957-60. To report and recommend on the duties, qualifications, and salaries for this office.

PLACE OF MEETING

Hubert Reisinger, East Orange, New Jersey, Chairman, 1958
Alice Hach, Racine, Wisconsin, 1957
Joseph N. Payne, Madison, Wisconsin, 1957
Forest N. Fisch, Greeley, Colorado, 1958
*Glenn Ayre, Macomb, Illinois, 1959
*Marguerite Brydegaard, San Diego, California, 1959

Duties: To select, request, and receive invitations to hold conventions and advise the Board on suitability of places for meeting. To continue to study the optimum number of meetings to be held each year.

POLICY, 1958

John Mayor, Madison, Wisconsin, Chairman
*Myrl Ahrendt, Washington, D. C.
*John R. Clark, New Hope, Pennsylvania
*Daniel Lloyd, Washington, D. C.
*William Zant, Stillwater, Oklahoma

Duties: To study and report general policy matters relating to Board and Council action. To study procedures for collecting funds and soliciting grants to aid research and study by the National Council.

PUBLICATION BOARD

*Clifford Bell, Los Angeles, California, Chairman, 1957
Ben Sueltz, *The Arithmetic Teacher*, Editor, 1957
Max Beberman, *The Mathematics Student Journal*, Editor, 1958
*Robert Fouch, Tallahassee, Florida, 1958 (Chairman, 1957-58)
Henry Swain, Supplementary Publications Editor, 1958
*Glenn Ayre, Macomb, Illinois, 1959 (Chairman, 1958-59)
Henry Van Engen, THE MATHEMATICS TEACHER, Editor, 1959
M. H. Ahrendt, Executive Secretary (ex-officio)

Duties: By motion of the Board this is the name given to a special committee on Publications and now made permanent. Its duties are outlined in the report submitted to the Board of Directors at Milwaukee in April, 1956, as follows:

A. The duties of the Publications Board should consist of:
   1. Co-ordinating activities and establishing policies for all publications of the Council.
   2. Making major recommendations to the Board concerning the publications of Council and the policies governing the same.
   3. Taking care of all minor publication details without the necessity of reporting for Board action. Any such minor actions taken would be reported to the President.
   4. Reading manuscripts submitted by the Chairman of the Supplementary Publications Committee for final acceptance or rejection. (It is assumed that the Chairman of the Supplementary Publications Committee would submit manuscripts that have some chance of being accepted. A manuscript that obviously is of no value as a supplementary publication should be rejected by the Supplementary Publications Committee.)

B. The relationship of editors and chairmen of publications committees to the Publications Board.
   1. All editors and chairmen of publications committees are immediately responsible to the Publications Board and are expected to clear all major decisions affecting their publications with the Publications Board.

PUBLICITY, 1958

James Zant, Stillwater, Oklahoma, Chairman
Myrl Ahrendt, Washington, D. C.
Eunice Lewis, Norman, Oklahoma
*Margaret Striegl, Wauwatosa, Wisconsin

Duties: To suggest policy and means of advertising the Council and its activities; how to publicize speeches, and convention reports; how to bring mathematics into favorable view of the public.

RELATIONS WITH THE NEA

Arlene Archer, Richmond, Virginia, Chairman, 1957
Josephine Berkey, Washington, D. C., 1957
Virginia Lee Pratt, Omaha, Nebraska, 1957
*Ida Mae Bernhard, Austin, Texas, 1958
*Julia E. Diggins, Washington, D. C., 1958
*William Gager, Gainesville, Florida, 1958
*Annie John Williams, Durham, North Carolina, 1958

Duties: To report on the interrelation, responsibilities of services of NCTM to NEA and vice versa and to suggest policy on these interrelations. To report on value of the organizations to the good of each other.

RESEARCH

*John Kinsella, New York, New York, 1959, Chairman, 1957

Maurice Hartung, Chicago, Illinois, 1957
Kenneth Brown, Washington, D. C., 1958
Nathan Lazar, Columbus, Ohio, 1958

Duties: To provide research section at convention; to propose and provide means of collecting and publishing research. To consider establishment of supplement to *Mathematical Reviews* to contain World Review Research in mathematics education.

SUPPLEMENTARY PUBLICATIONS

Henry Swain, Winnetka, Illinois, 1958, Chairman
Houston Banks, Nashville, Tennessee, 1957
Lawrence Bartnick, Natick, Massachusetts, 1957
Rachel Keniston, Stockton, California, 1957
Vera Sanford, Oneonta, New York, 1957
Marguerite Brydegaard, San Diego, California, 1958
Kenneth Kidd, Gainesville, Florida, 1958
Jesse Osborne, St. Louis, Missouri, 1958
Dwain Small, Carbondale, Illinois, 1958
*Margaret Josephs, Milwaukee, Wisconsin, 1959
*Lawrence A. Ringenberg, Charleston, Illinois, 1959
*Robert Seber, Kalamazoo, Michigan, 1959

Duties: To recommend for publication, edit, and produce brochures, pamphlets, leaflets, etc., of interest to teachers of mathematics.

TEACHER EDUCATION, CERTIFICATION AND RECRUITMENT

(Chairman to be appointed)
Charles R. Atherton, Shepherdstown, West Virginia, 1957
Kenneth Brown, Washington, D. C., 1957
Robert Kalin, Princeton, New Jersey, 1957
David Page, Urbana, Illinois, 1958
Richard Purdy, San Jose, California, 1958
Myron F. Rosskopf, New York, New York, 1958

Duties: To study the role the National Council shall play in determining the character and quality of mathematics instruction; to propose ways and means of making the Council effective in protecting the future of our profession by advisement on education and certification, etc.

*TELEVISION

Lewis Scholl, Buffalo, New York, Chairman, 1959
Philip Peak, Bloomington, Indiana, 1957
Robert Salsburg, Grand Rapids, Michigan, 1957
Phillip Jones, Ann Arbor, Michigan, 1958
Edward S. Sherley, Schenectady, New York, 1958
Sylvia Vopni, Seattle, Washington, 1959

Duties: To study procedures by which the Council can put mathematics into television.

The Arithmetic Teacher

Ben A. Sueltz, Cortland, New York, Editor, 1957
Marguerite Brydegaard, San Diego, California, Associate Editor
John R. Clark, New Hope, Pennsylvania, Associate Editor

The Mathematics Student Journal

Max Beberman, Urbana, Illinois, Editor, 1958
L. J. Adams, Santa Monica, California, Associate Editor
Izaak Wirszup, Chicago, Illinois, Associate Editor

THE MATHEMATICS TEACHER

Henry Van Engen, Editor, 1959
Jackson B. Adkins, Exeter, New Hampshire
Mildred Keiffer, Cincinnati, Ohio
Z. L. Loflin, Lafayette, Louisiana
Philip Peak, Bloomington, Indiana
Ernest Ranucci, Newark, New Jersey
M. F. Rosskopf, New York, New York

YEARBOOK PLANNING

George Hawkins, LaGrange, Illinois, Chairman, 1958
Harold Fawcett, Columbus, Ohio, 1957
*M. F. Rosskopf, New York, New York, 1959

Duties: To report on the present status of yearbooks; to recommend further areas for possible yearbooks, especially one on students not going to college; to recommend ways of procedure for planning yearbooks.

YEARBOOKS

23*rd* (1957) *Foundations of Mathematics*
F. Lynwood Wren, Nashville, Tennessee, Chairman
Carl B. Allendoerfer, Seattle, Washington
Bruce Meserve, Montclair, New Jersey
Saunders MacLane, Chicago, Illinois
Carroll V. Newson, New York, New York

24*th* (1958) *Mathematical Concepts*
Phillip Jones, Ann Arbor, Michigan, Chairman
Harold Fawcett, Columbus, Ohio
Alice Hach, Racine, Wisconsin
Charlotte Junge, Detroit, Michigan
Henry W. Syer, Boston, Massachusetts
Henry Van Engen, Cedar Falls, Iowa

25*th* (?) *Evaluation*
Maurice Hartung, Chicago, Illinois, Chairman
Frank Allen, LaGrange, Illinois
Donovan Johnson, Minneapolis, Minnesota
Robert Pingry, Champaign, Illinois
Fred J. Weaver, Boston, Massachusetts

26*th* (?) *Arithmetic*
Foster Grossnickle, Jersey City, New Jersey, Chairman
Dan Dawson, Stanford, California
Ida Mae Heard, Lafayette, Louisiana
Irene Sauble, Detroit, Michigan
Herbert Spitzer, Iowa City, Iowa
Louis C. Thiele, Detroit, Michigan

Duties: To secure manuscript and supervise the publication of yearbooks in selected areas.